

DUAL BASES FUNCTIONS IN SUBSPACES

SCOTT N. KERSEY

ABSTRACT. In this paper we study dual bases functions in subspaces. These are bases which are dual to functionals on larger linear space. Our goal is construct and derive properties of certain bases obtained from the construction, with primary focus on polynomial spaces in B-form. When they exist, our bases are always affine (not convex), and we define a symmetric configuration that converges to Lagrange polynomial bases. Because of affineness of our bases, we are able to derive certain approximation theoretic results involving quasi-interpolation and a Bernstein-type operator.

In a broad sense, it is the aim of this paper to present a new way to view approximation problems in subspaces. In subsequent work, we will apply our results to dual bases in subspaces of spline and multivariate polynomial spaces, and apply this to the construction of blended function approximants used for approximation in the sum of certain tensor product spaces.

1. INTRODUCTION

Let X be a finite dimensional vector spaces (of dimension n). Of fundamental importance is the basis. There are various reasons to choose a particular basis, and each basis has advantages and disadvantages. Often it is the action of certain functionals that lends importance to a particular basis. For example, the Lagrange basis is important because it is dual to point evaluation. In general, any basis $\Phi^n = [\Phi_1, \dots, \Phi_n]$ for X has a dual map $\Lambda^n = [\lambda_1, \dots, \lambda_n]$ for X^* satisfying $\lambda_i \Phi_j = \delta_{ij}$. That is, $\Lambda^n \Phi^n = I$, the identity matrix. Conversely, any basis $\Lambda^n \subset X^*$ is dual to some basis Φ^n of X . It is the dual map Λ^n that extracts information from the functions that is of particular interest here, which arguably plays a more prominent role than the basis Φ^n itself. This process allows one to consider the *information* we are trying to capture as our primary goal.

The above concepts are well-known and well-studied. In this paper, we are interested in investigating bases for *subspaces* Y of X that are dual to *subsets* of the functionals in the map Λ^n . For example, suppose that $\Lambda^n = [\lambda_1, \dots, \lambda_n]$ are linearly independent on the n -dimensional space X , and Y is an m -dimensional subspace of X with $m < n$. Then, given an injective map $s : [1 : m] \rightarrow [1 : n]$, our questions are:

- (1) Is the subset $\Lambda_n(s) = [\lambda_{s(1)}, \dots, \lambda_{s(m)}]$ of Λ_n linearly independent on Y ?
- (2) If linearly independent, what is the basis D^m for Y that is dual to $\Lambda^n(s)$ in the sense that $\Lambda^n(s)^T D^m = I$.
- (3) What are the properties of this dual basis.

Date: June 26, 2014.

1991 *Mathematics Subject Classification.* 41A05, 41A10, 41A15, 65D05 65D07.

In this paper we consider the action of subsets of functionals for certain basis on *subspaces* of the original space. This allows us to view how the subspace looks according to information on the whole space. In some sense, we are addressing the question “how do you approximate with less (or not enough) information?” But it is not always possible to construct such bases, because the functionals are not always *linearly independent* on the subspace. For example, the only subset of $\Lambda = [\delta_0, \delta_0 D, \delta_0 D^2]$ that is linearly independent on $\text{Ran}[1, (\cdot)]$ is $\Lambda = [\delta_0, \delta_0 D]$, not $\Lambda = [\delta_0, \delta_0 D^2]$ or $\Lambda = [\delta_0 D, \delta_0 D]$. As we will see in this paper, the situation is different in the Bernstein basis.

It is the objective of this paper to determine if dual bases exist for certain subspaces of polynomial spaces, and if so to compute and characterize these bases, and then determine their properties. In particular, we show that in the Bernstein basis for \mathbb{S}_n , any $m+1$ -selection of the dual functionals are linearly independent on \mathbb{S}_m . Moreover, by choosing a particular symmetric choice of the functionals, we show that the corresponding dual basis converge to the Lagrange polynomial basis. Later, we derive certain approximation results concerning quasi-interpolation and a Bernstein-type operator.

2. DUAL BASES IN SUBSPACES Y OF X

As defined above, X is a vector space of dimension n with basis Φ^n and dual map Λ^n , and Y a subspace of X of dimension m , $m < n$, with basis Φ^m . For $y \in Y \subset X$, we have $y = \Phi^m \alpha = \Phi^n \beta$ for some coefficient sequences $\alpha \in \mathbb{R}^m$ and $\beta \in \mathbb{R}^n$. Applying the dual basis, we have

$$\Lambda^{nT} \Phi^m \alpha = \Lambda^{nT} \Phi^n \beta = \beta.$$

Hence, $\beta = E\alpha$ with $E := \Lambda^{nT} \Phi^m$. The matrix E embeds the coefficients α of y in the basis for Y to it's coefficients in the basis for X . By inclusion of Y into X , we can construct the embedding $e := \Phi^n \circ E \circ (\Phi^m)^{-1}$ of Y into X . And since $\Phi^m \alpha = \Phi^n E\alpha$ for all $\alpha \in \mathbb{R}^m$, it follows that $\Phi^m = \Phi^n E$. All said, this can be visualized as in the following commutative diagram.

$$\begin{array}{ccc} Y & \xrightarrow{e} & X \\ \Phi^m \uparrow & & \uparrow \Phi^n \\ \mathbb{R}^m & \xrightarrow{E} & \mathbb{R}^n \end{array}$$

FIGURE 2.1. Embedding of $\mathbb{S}_{m,y}$ into $\mathbb{S}_{n,x}$.

Moreover, we note that

$$\Lambda^{nT} \Phi^m = \Lambda^{nT} \Phi^n E = E = E \Lambda^{mT} \Phi^m = (\Lambda^m E^T)^T \Phi^m.$$

Therefore, $\Lambda^n = \Lambda^m E^T$. Hence, we have the following result concerning the action of Λ^n on a subspace Y of X .

Proposition 2.1. *Let (Y, Φ^m, Λ^m) be an m -dimensional subspace of a vector space (X, Φ^n, Λ^n) , with basis Φ^m dual to Λ^m and Φ^n dual to Λ^n . Then, Y embeds into X by the map $e = \Phi^n \circ E \circ (\Phi^m)^{-1}$*

with $E = \Lambda^{nT} \Phi^m$, and the bases transform as

$$\boxed{\Phi^m = \Phi^n E}$$

Further, the dual basis for X^* map to the dual basis for Y^* by the map

$$\boxed{\Lambda^n = \Lambda^m E^T}$$

Since $\Lambda^n \subset X^* \subset Y^*$, the functionals $\lambda_i^n \in X^*$ are also functionals on Y . But since $\dim Y < \#\Lambda^n$, the functionals in Λ^n cannot be linearly independent on Y . However, since

$$\Lambda^{nT} \Phi^m = \Lambda^{nT} \Phi^n E = E$$

is a 1-1 matrix, Λ^n does span Y^* (moreover, $\Lambda^n E$ is a basis for Y^* since $(\Lambda^n E)^T \Phi^m = E^T E$ is invertible). Therefore, we can trim Λ^n to a basis for Y^* by removing *inessential* vectors (this is a standard construction, c.f. [5]). This leaves an m -subvector $\Lambda^n(s)$ of Λ^n that is linearly independent on Y , where s is an injective map $s : [1 : m] \rightarrow [1 : n]$ (which we call a *selection map*). Then, with I the $n \times n$ identity matrix, we have

$$\Lambda^n(s)^T \Phi^m = \Lambda^n(s)^T \Phi^n E = (\Lambda^n I(:, s))^T \Phi^n E = I(s, :) \Lambda^{nT} \Phi^n E = I(s, :) E = E(s, :).$$

Therefore, an m -subvector $\Lambda^n(s)$ of Λ^n is linearly independent on Y iff $E(s, :)$ is invertible. Moreover, when $E(s, :)$ is invertible, the basis of $Y = \text{Ran}(\Phi^m)$ that is dual to $\Lambda^n(s)$ is $\Phi^m E(s, :)^{-1}$, as follows by computing the change of basis $\Phi^m \rightarrow \Phi^m A$:

$$I = \Lambda^n(s)^T \Phi^m A = E(s, :) A.$$

Hence, the basis of Y that is dual to $\Lambda^n(s)$ is $D^m := \Phi^m E(s, :)^{-1}$. The following summarizes the above statements:

Proposition 2.2. *Let (X, Φ^n, Λ^n) be a vector space with basis Φ^n dual to Λ^n , and let (Y, Φ^m, Λ^m) be a subspace with basis $\Phi^m = \Phi^n E$ dual to Λ^m . Let $s : [1 : m] \rightarrow [1 : n]$ denote a selection (injective) map.*

- (1) $\Lambda^n E$ is linearly independent on Y and of length m (hence a basis for Y^*).
- (2) There exist (at least one) selections $\Lambda^n(s)$ of Λ^n linearly independent on Y . Sometimes only one selection is linearly independent, and sometimes all selections are linearly independent (this is of particular interest later in this paper).
- (3) $E(s, :) = \Lambda^n(s)^T \Phi^m$
- (4) $\Lambda^n(s)$ is linearly independent on Y iff $E(s, :)$ is invertible.
- (5) If $\Lambda^n(s)$ is linearly independent on Y , then the basis of Y that is dual to $\Lambda^n(s)$ in the sense that $\Lambda^n(s)^T D^m = I$ is $D^m = \Phi^m E(s, :)^{-1}$.

3. ADDITIONAL PROPERTIES OF DUAL BASES IN SUBSPACES

In this section we state definitions and properties useful for certain constructions that we consider in the subsequent sections. Firstly, by Proposition 2.2, there always exists some s such that $\Lambda^n(s)$ is linearly independent on Y , and in this case we have the basis $D^m = \Phi^m E(s, \cdot)^{-1}$ with $E = \Lambda^n(s)^T \Phi^m$. However, this may or may not be true for all s . Hence, we give the following definition to make this distinction.

Definition 3.1. We say the embedding of (Y, Φ^m, Λ^m) into (X, Φ^n, Λ^n) is *complete* if $\Lambda^n(s)$ is linearly independent on Y for all injective maps (selections) $s : [1 : m] \rightarrow [1 : n]$.

Proposition 3.2. *Let $Y = \mathbb{S}^m$ be the space of polynomials of degree at most m , and let $X = \mathbb{S}^n$ be the space of polynomials of degree at most n .*

- (1) *Let Φ^m and Φ^n be power basis for \mathbb{S}^m and \mathbb{S}^n . Then, the embedding is not complete. Moreover, $\Lambda^n(s)$ is linearly independent on Φ^m iff $s = [0 : m]$.*
- (2) *Let Φ^m and Φ^n be Bernstein bases for \mathbb{S}^m and \mathbb{S}^n , respectively. Then, the embedding is complete.*

Proof. Part (2) is non-trivial, and will be proved later in this paper. For part (1), recall that the power basis $P^n = [1, (\cdot), \dots, (\cdot)^n]$ dual to $\Lambda^n = [\delta_0, \delta_0 D, \dots, \delta_0 D^n/n!]$, and $(\mathbb{S}_m, P^m, \Lambda^m)$ is the subspace with $m < n$. Then $P^m = P^n E$ for $E = I(:, 0 : m)$, with I is the $(n+1) \times (n+1)$ identity matrix. It follows by Proposition 2.1 that $\Lambda^n = \Lambda^m E^T = \Lambda^m I(0 : m, \cdot)$ on \mathbb{S}_m . Let s be a selection map. Since $I(0 : m, s)$ is invertible iff $s = 0 : m$, it follows that $\Lambda^n(s)$ is linearly independent on \mathbb{S}_m iff $s = 0 : m$. That is, on \mathbb{S}_m , $[\lambda_0^n, \dots, \lambda_m^n] = [\lambda_0^m, \dots, \lambda_m^m]$. Hence, there is only one choice for the selection s of Λ^n to consider, and this choice is $\Lambda^m = \Lambda^n(s)$. Hence, the Lagrange basis is not complete. \square

The next properties are characteristic of Bernstein bases, which we consider later in this paper.

Definition 3.3.

- (1) Let Φ^n be a basis of finite-dimensional function space. We say Φ^n is *affine on S* if $\Phi^n(t)\alpha$ is an affine combination of α for all $t \in S$. That is, $\sum_i \Phi_i^n(t) = 1$ for all $t \in S$. If it holds for all $t \in \text{dom}(\Phi^n)$, then we say Φ^n is *affine*.
- (2) Let E be a (real) matrix. We say E is *row affine* if $\sum_j E(i, j) = 1$ for all i , and *column affine* if $\sum_i E(i, j) = 1$ for all j .

Lemma 3.4.

- (1) *Matrix inversion preserves the property row-affine. That is, the inverse of an invertible row-affine matrix is row-affine.*
- (2) *Matrix multiplication preserves the property row-affine. That is, the product of two row-affine matrices is row-affine.*

Proof. For the first result, let A be an $n \times n$ invertible row-affine matrix. Then $\sum_{j=1}^n A(i, j) = 1$ for $i = 1 : n$. Since $A^{-1}A = I$,

$$1 = \sum_{j=1}^n I(k, j) = \sum_{j=1}^n \sum_{i=1}^n A^{-1}(k, i)A(i, j) = \sum_{i=1}^n A^{-1}(k, i) \sum_{j=1}^n A(i, j) = \sum_{i=1}^n A^{-1}(k, i),$$

for $k = 1 : n$. Hence, A^{-1} is row-affine.

For the second result, assume $C = AB$ with A and B row-affine of dimensions $m \times n$ and $n \times p$, respectively. Then, C is of dimension $m \times p$, and

$$\sum_{j=1}^p C(k, j) = \sum_{j=1}^p \sum_{i=1}^n A(k, i)B(i, j) = \sum_{i=1}^n A(k, i) \sum_{j=1}^p B(i, j) = \sum_{i=1}^n A(k, i) = 1,$$

for $k = 1 : m$. Therefore, C is row-affine. \square

Theorem 3.5. *Let Φ^m and Φ^n be affine bases of Y and X , respectively, with Y a subspace of X .*

- (1) *If $\Phi^m = \Phi^n E$ for some matrix E , then E is row affine.*
- (2) *If, moreover, $E(s, \cdot)$ is invertible for some selection $s : [1 : m] \rightarrow [1 : n]$, then the dual basis $D^m = \Phi^m E(s, \cdot)^{-1}$ exists and is affine.*

Proof. For (1), it follows by affineness of the two bases that

$$1 = \sum_i \Phi_i^m(t) = \sum_i \sum_j \Phi_j^n(t)E(j, i) = \sum_j \Phi_j(t) \sum_i E(j, i).$$

Let λ_k^n be the functional on X such that $\lambda_k \Phi_j^n = \delta_{ij}$. Then,

$$\lambda_k^n(1) = \lambda_k^n\left(\sum_j \Phi_j^n\right) = \sum_j \lambda_k^n(\Phi_j^n) = \lambda_k^n(\Phi_k^n) = 1,$$

and so

$$1 = \lambda_k^n(1) = \lambda_k^n\left(\sum_j \Phi_j(t) \sum_i E(j, i)\right) = \sum_i E(k, i).$$

This establishes (1).

For (2), we recall that $D^m = \Phi^m E(s, \cdot)^{-1}$ is the basis for Y dual to $\Lambda^n(s)$ with $E = \Lambda^{nT} \Phi^m$, and hence $E(s, \cdot) = \Lambda^n(s)^T \Phi^m$. By part (1) of this theorem, $E(s, \cdot)$ is row affine, and by Lemma 3, $A := (E(s, \cdot))^{-1}$ is row affine as well. Therefore,

$$\sum_{i=0}^n D_i^m = \sum_{i=0}^m \sum_{j=0:m} \Phi_j^m A(j, i) = \sum_{j=0}^m B_j^m \sum_{i=0:m} A(j, i) = \sum_{j=0}^m B_j^m = 1.$$

And so D^m is an affine basis. \square

These results regarding affine bases will concern the Bernstein basis, which we investigate in the remaining sections, and in particular we show that dual bases are affine. We remark here that the same is not true of convexity. That is, the dual bases constructed are not convex. Indeed, matrix inversion does not preserve convexity, as it does affineness. Moreover, as it turns out, certain approximation properties do not actually require convexity. I.e., affineness is enough. Hence, our

dual bases D^m will enjoy many of the same properties as B^m do, in the Bernstein-basis setting, just not convexity.

In our construction we derive dual bases in terms of data maps that are dual on a large space. However, it is possible to have different data maps that are both dual to the same bases. For example, in terms of the point-evaluation function $\delta_x f = f(x)$ and derivative operator D , both maps $\Lambda = [\delta_0, \delta_0 D]$ and $\tilde{\Lambda} = [\delta_0, \delta_1 - \delta_0]$ are dual to the power basis $\Phi = [1, (\cdot)]$, as can be seen by $\Lambda^T \Phi = \tilde{\Lambda}^T \Phi = I$. However, even with different data maps, dual bases are the same, as shown next. This idea is important in constructing approximation operators where some data maps may apply and not others. More to the point, we will derive dual basis in this paper with respect to dual data maps that involve differentiation, hence do not apply on $C([0, 1])$. In the last section of this paper, we define new data maps that do not involve differentiation to derive certain properties of approximation operators on $C([0, 1])$.

Proposition 3.6. *Let Λ^n and $\tilde{\Lambda}^n$ be data maps on X both dual to the basis Φ^n . Then both maps are equivalent on X , and the dual bases in a subspace Y are invariant of which dual map is used. That is, if D^m and \tilde{D}^m are bases for the subspace Y that are dual to Φ^m with respect to the selection s and data maps Λ^n and $\tilde{\Lambda}^n$, respectively. Then, $D^m = \tilde{D}^m$.*

Proof. To show that Λ^n and $\tilde{\Lambda}^n$ are equivalent on X , let $f = \Phi^n \alpha \in X$. Then, since both maps are dual to Φ^n , we have $\Lambda^{nT} \Phi^n \alpha = \alpha$ and $\tilde{\Lambda}^{nT} \Phi^n \alpha = \alpha$. Hence, they are equivalent on X .

The basis D^m is dual to $\Lambda^n(s)$ in the sense that $\Lambda^{nT} D^m = I$. With this basis represented $D^m = \Phi^m A = \Phi^n E A$, for some embedding, we get

$$\Lambda^{nT} D^m = \Lambda^{nT} \Phi^n E A = E A.$$

Therefore, $I = \Lambda^{nT}(s) D^m = E(s, :) A$, and so $A = E(s, :)^{-1}$, which depends only on the embedding E and not the data map Λ^n . Hence, if \tilde{A} is the transformation matrix for $\tilde{\Lambda}^n$, then $\tilde{A} = A$, and so $D^m = \tilde{D}^m$. \square

4. DUAL BERNSTEIN BASES IN THE SUBSPACE \mathbb{S}_m OF \mathbb{S}_n

Let \mathbb{S}_n be the space of polynomials of degree at most n and \mathbb{S}_m the subspace of polynomials of degree at most m , for $m < n$. In this section we derive bases for \mathbb{S}_m that are dual to subsets of the dual Bernstein functionals on \mathbb{S}_n . We begin with some basic formulas involving Bernstein polynomials that will be used.

- Bernstein basis for \mathbb{S}_n : $B^n = [B_0^n, \dots, B_n^n]$ with $B_i^n = \binom{n}{i} (1 - \cdot)^{n-i} (\cdot)^i$, $\sum_{i=0}^n B_i^n = 1$ and $B_i^n \geq 0$ on $[0, 1]$. That is, it forms a partition of unity on $[0, 1]$. In particular, $B^n \alpha$ is a convex combination of α for all $t \in [0, 1]$.
- Degree Elevation of Bernstein Basis (See [1, 6]): $B^m = B^n E$ with E the $(n + 1) \times (m + 1)$ matrix with entries

$$E(i, j) = \frac{\binom{n-i}{m-j} \binom{i}{j}}{\binom{n}{m}} = \frac{\binom{n-m}{i-j} \binom{m}{j}}{\binom{n}{i}}$$

for $0 \leq i \leq n$ and $0 \leq j \leq m$, with $E(i, j) = 0$ if $i < j$ or $n - i < m - j$. That is,

$$B_j^m = \sum_{i=0}^n E(i, j) B_i^n = \sum_{i=0}^n \frac{\binom{n-i}{m-j} \binom{i}{j}}{\binom{n}{m}} B_i^n.$$

- Dual Bernstein Functionals: $\Lambda^n = [\lambda_0^n, \dots, \lambda_n^n]$ with

$$\lambda_k^n = \sum_{j=0}^k \frac{\binom{k}{j}}{\binom{n}{j}} \frac{1}{j!} \delta_0 D^j = \sum_{j=0}^{n-k} (-1)^j \frac{\binom{n-k}{j}}{\binom{n}{j}} \frac{1}{j!} \delta_1 D^j.$$

Therefore, $\lambda_k^n B_i^n = \delta_{ki}$ and $\Lambda^{nT} B^n = I$. (Note that the two forms are equivalent on \mathbb{S}_n , but not on all spaces.)

- Reduction of Dual Bernstein Basis: On \mathbb{S}_m , $\Lambda^n = \Lambda^m E^T$ (by Proposition 2.1). Hence,

$$\lambda_k^n = \sum_{j=0}^m E^T(j, k) \lambda_j^m = \sum_{j=0}^m \frac{\binom{n-k}{m-j} \binom{k}{j}}{\binom{n}{m}} \lambda_j^m.$$

Hence, in the context of this paper, we have the following:

Theorem 4.1. *Let B^m be the Bernstein basis for \mathbb{S}^m , and let B^n be the Bernstein basis for \mathbb{S}^n with dual map Λ^n given above, $m \leq n$. Let E be the degree elevation matrix. Then,*

- (1) *The embedding e of $(\mathbb{S}^m, B^m, \Lambda^m)$ into $(\mathbb{S}^n, B^n, \Lambda^n)$ is complete.*
- (2) *The basis for \mathbb{S}^m dual to $\Lambda^n(s)$ can be represented $D^m = B^m E(s, \cdot)^{-1}$, with $D^m = B^m$ when $m = n$.*

Proof. For (1), we need to show that $\Lambda^n(s)$ is linearly independent on \mathbb{S}^m for any selection $s : [0 : m] \rightarrow [0 : n]$. The conversion from between the power basis P^n and Bernstein basis B^n for \mathbb{S}_n can be expressed

$$P^n D_n = B^n T_n$$

with $D_n := \text{Diag}([\binom{n}{j} : j = 0 : n])$ and T_n Pascal's (lower triangular) matrix

$$T^n := \left[\binom{i}{j} : 0 \leq i, j \leq n \right] = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n} \end{bmatrix}.$$

This follows directly from the identity

$$\binom{n}{j} t^j = \sum_{i=0}^n \binom{i}{j} B_i^n$$

(see [6], section 2.8, for a short proof). Let $d^n := (D_n)^{-1} = [1/\binom{n}{j} : j = 0 : n]$. Then,

$$\begin{aligned}\Lambda^n(s)^T P^m &= (\Lambda^n I(:, s))^T P^n I(:, 0 : m) \\ &= I(s, :) \Lambda^{nT} B^n T^n d^n I(:, 0 : m) \\ &= I(s, :) T^n d^n I(:, 0 : m) \\ &= T^n(s, 0 : m) d^n(0 : m, 0 : m)\end{aligned}$$

In [3] it was shown that the truncated Pascal matrix $T^n(s, 0 : m)$ is invertible for any $m+1$ selection $s : [0 : m] \rightarrow [0 : n]$ of the rows of T^n . Therefore, $\Lambda^n(s)^T P^m$ is invertible, and so $\Lambda^n(s)$ is linearly independent on \mathbb{S}_m .

For (2), we have $B^m = B^n E$ with E the degree elevation matrix. Therefore, we are exactly in the framework of Proposition 2.2, with $\Phi^m = B^m$ and $\Phi^n = B^n$. Since the embedding is complete, we have that $E(s, :)$ is invertible for any selection map s , Therefore, the dual basis of \mathbb{S}^m dual to $\Lambda^n(s)$ exists for any selection map s , and can be represented $D^m = D^m = B^m E(s, :)^{-1}$. In the case $m = n$, there is only one selection $s = [1 : n]$, and so $\Lambda^n(s) = \Lambda^n$. Hence, $E(s, :) = E$ is the identity matrix, and $D^m = B^m = B^n$ are therefore dual to Λ^n . \square

Since, as is well known, B^m and B^n are affine bases, the next result follows directly from Theorem 3.5. Instead, we give a more direct proof below using properties of the degree elevation matrix E .

Theorem 4.2. *For any selection (injective) map $s : [0 : m] \rightarrow [0 : n]$, $D^m := B^m A$ is an affine basis of \mathbb{S}_m dual to $\Lambda^n(s)$, with $A := E(s, :)^{-1}$.*

Proof. We note that by the previous lemma $\Lambda^n(s)$ is linearly independent on \mathbb{S}_m . Therefore, $\Lambda^n(s)^T B^m$ is invertible. By Proposition 2.2, $\Lambda^n(s)^T B^m = E(s, :)$ and $D^m = B^m E(s, :)^{-1}$ is the basis for \mathbb{S}_m dual to $\Lambda^n(s)$. Recall that

$$E(i, j) = \frac{\binom{n-i}{m-j} \binom{i}{j}}{\binom{n}{m}}.$$

By the Chu-Vandermonde identity

$$\binom{n}{m} \sum_{j=0}^m E(i, j) = \sum_{j=0}^m \binom{n-i}{m-j} \binom{i}{j} = \sum_{j=0}^m \binom{n-i}{j} \binom{i}{m-j} = \binom{n}{m},$$

and so $E(s, :)$ is row affine. By Lemma 3, $(E(s, :))^{-1}$ is row affine as well. Therefore,

$$\sum_{i=0}^n D_i^m = \sum_{i=0}^m \sum_{j=0:m} B_j^m A(j, i) = \sum_{j=0}^m B_j^m \sum_{i=0:m} A(j, i) = \sum_{j=0}^m B_j^m = 1.$$

And so D^m is an affine basis: \square

The next property is a generalization of the following property of Bernstein polynomials:

$$x = B^m(x) v^m = \sum_{i=0}^m \xi_i^m B_i^m(x)$$

for all x , with $\xi_i^m := \frac{i}{m}$.

Theorem 4.3. *Let $D^m = B^m E(s, \cdot)^{-1}$ be the dual basis for some selection map s .*

- (1) $E\xi^m = \xi^n$
- (2) $\xi^m = E(s, \cdot)^{-1}\xi^n(s)$.
- (3) $x = B^m\xi^m = D^m(x)\xi^n(s) = \sum_{i=0}^m \xi_{s(i)}^n D_i^m(x)$.

Proof. Recall that the Bernstein functions on \mathbb{S}^n can be represented

$$\lambda_k^n = \sum_{j=0}^k \frac{\binom{k}{j}}{\binom{n}{j}} \frac{1}{j!} \delta_0 D^j.$$

In particular, $\lambda_0^n x = \delta_0 x = 0$. For $k > 0$, the sum of the terms for $\lambda_k^n x$ are zero except when $j = 1$. Hence,

$$\lambda_k^n x = \sum_{j=0}^n \frac{\binom{k}{j}}{\binom{n}{j}} \frac{1}{j!} \delta_0 D^j x = \frac{\binom{k}{1}}{\binom{n}{1}} \frac{1}{1!} \delta_0 D^1 x = \frac{k}{n}.$$

Therefore, $\Lambda^{nT} x = \xi^n$. From this we get,

$$\xi^n = \Lambda^{nT} x = \Lambda^{nT} B^m(x)\xi^m = \Lambda^{nT} B^n(x)E\xi^m = E\xi^m.$$

Therefore, $\xi^n = E\xi^m$. Hence, $\xi^n(s) = E(s, \cdot)\xi^m$ implies $\xi_m = E(s, \cdot)^{-1}\xi^n(s)$, and

$$D^m(x)\xi^n(s) = B^m(x)E(s, \cdot)^{-1}\xi^n(s) = B^m(x)\xi_m = x.$$

□

5. PLOTTING POLYNOMIALS IN THE DUAL BASES

Since dual bases D^m are bases for the subspace Y , and function in Y can be represented by this basis. In particular, in the Bernstein setup above, any polynomial $p \in S^m$ can be represented as $p = D^m\alpha$ for some $\alpha \in \mathbb{R}^{n+1}$. And so, this basis can be used in computation with functions in this polynomial space.

Now, since $D^m = B^m A$ with $A = E(s, \cdot)^{-1}$, we can write $p = B^m(A\alpha)$. Hence, one can transform the coefficients α by the matrix A , and then use B-form techniques in computation. In particular, to plot the curves, one can use DeCasteljau's algorithm on the control polygon with points $(\frac{i}{m}, (A\alpha)(i))$, for $i = 0 : m$.

This is depicted in Figure 5.1. The control polygon for the coefficients α is displayed in solid broken line and the transformed control polygon is dashed.

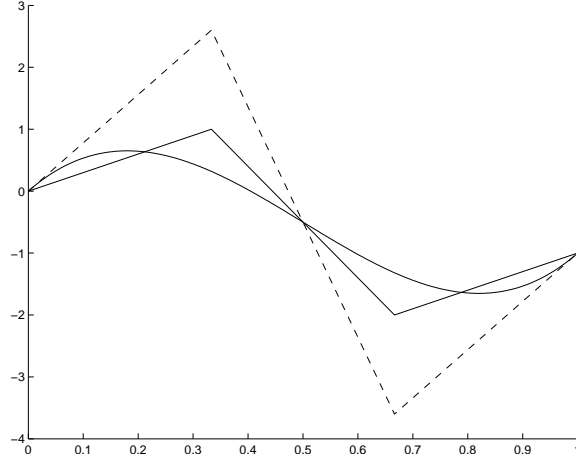


FIGURE 5.1. Control polygon and transformed polygon for deCasteljau's algorithm with dual Bernstein subspace bases.

6. SYMMETRIC BERNSTEIN CLASS OF DUAL BASES

As shown in the previous section, for Bernstein functionals the matrix $E(s, :) = \Lambda(s)^{nT} B^m$ is invertible for any selection map s , and we can therefore find dual bases for any selection map. Moreover, these bases are affine. In this section we will use this idea to produce a certain “symmetric” class of bases, and show that these converge to the Lagrange polynomial basis. We also provide an estimate for the rate of convergence.

For $k \in \mathbb{N}$, let $s(i) = ik$ for $i = 0 : m$. Hence, $s = 0 : k : n$ with $n = k * m$. For example:

- If $k = 1$ then $m = m$ and we get $s = [0 : m]$.
- If $m = 4$ and $k = 3$, then $n = 12$ and $s = [0, 3, 6, 9, 12]$.

The dual Bernstein bases are then $D_k^m = B^m A_{m,k}$ with $A_{m,k} = E(s, :)^{-1}$. The goal in the remainder of this section is to show that symmetric dual bases converge to point evaluation in a certain sense. To get the most general result, we extend the Bernstein functional to allow $k = x$ to be any real number:

$$\lambda_x^n := \sum_{j=0}^{\lfloor |x| \rfloor} \binom{x}{j} \frac{1}{j!} \delta_0 D^j.$$

Here, the factorials in the binomial coefficients will involve the gamma function when x is a non-integer. The functionals reduce to the above formulation when $k := x$ is a non-negative integer with $k \leq n$. For the following, we use the *falling factorial* notation

$$(x)_j := x \cdot (x - 1) \cdots (x - j + 1).$$

Lemma 6.1. For $x \in \mathbb{R}$ and $j \geq 0$,

$$\lim_{n \rightarrow \infty} \frac{\binom{xn}{j}}{\binom{n}{j}} = x^j,$$

with $(\cdot)! := \Gamma(\cdot + 1)$ for non-integer factorials.

Proof. By the well-known property $\Gamma(z + 1) = z\Gamma(z)$, we get that

$$j! \binom{xn}{j} = \frac{\Gamma(xn + 1)}{\Gamma(xn - j + 1)} = \frac{(xn)_j \Gamma(xn - j + 1)}{\Gamma(xn - j + 1)} = (xn)_j.$$

Therefore,

$$\lim_{n \rightarrow \infty} \frac{\binom{xn}{j}}{\binom{n}{j}} = \lim_{n \rightarrow \infty} \frac{xn}{n} \frac{xn - 1}{n - 1} \cdots \frac{xn - j + 1}{n - j + 1} = x^j.$$

□

Proposition 6.2. *Let $x \in \mathbb{R}$ and $p \in \mathcal{S}_m$ for some m . The dual functionals converge to point evaluation in the following sense*

$$\lim_{n \rightarrow \infty} \lambda_{xn}^n p = p(x).$$

In particular, with $n = mk$ and $x = \frac{i}{m}$,

$$\lim_{k \rightarrow \infty} \lambda_{ik}^{km} p = p\left(\frac{i}{m}\right).$$

Proof. Let $p(x) = \sum_{j=0}^m \alpha_j x^j$ in \mathcal{S}_m . Then, $p^{(j)}(0) = j! \alpha_j$ if $j \leq m$ and 0 otherwise, and so by the previous lemma we have

$$\lim_{n \rightarrow \infty} \lambda_{xn}^n p = \lim_{n \rightarrow \infty} \sum_{j=0}^{\lfloor xn \rfloor} \frac{\binom{xn}{j}}{\binom{n}{j}} \frac{1}{j!} p^{(j)}(0) = \lim_{n \rightarrow \infty} \sum_{j=0}^m \frac{\binom{xn}{j}}{\binom{n}{j}} \alpha_j = \sum_{j=0}^m \alpha_j \lim_{n \rightarrow \infty} \frac{\binom{xn}{j}}{\binom{n}{j}} = \sum_{j=0}^m \alpha_j x^j = p(x).$$

□

Corollary 6.3. *Let $s = 0 : k : km$ and $D_k^m = B^m A_k$ with $A_k = E(s, \cdot)^{-1}$. Then, $D_k^m \rightarrow L^m$ with L^m the Lagrange basis.*

In figure 6.1, we display the dual bases for various degree polynomial spaces and level of refinement to illustrate convergence to the Lagrange basis.

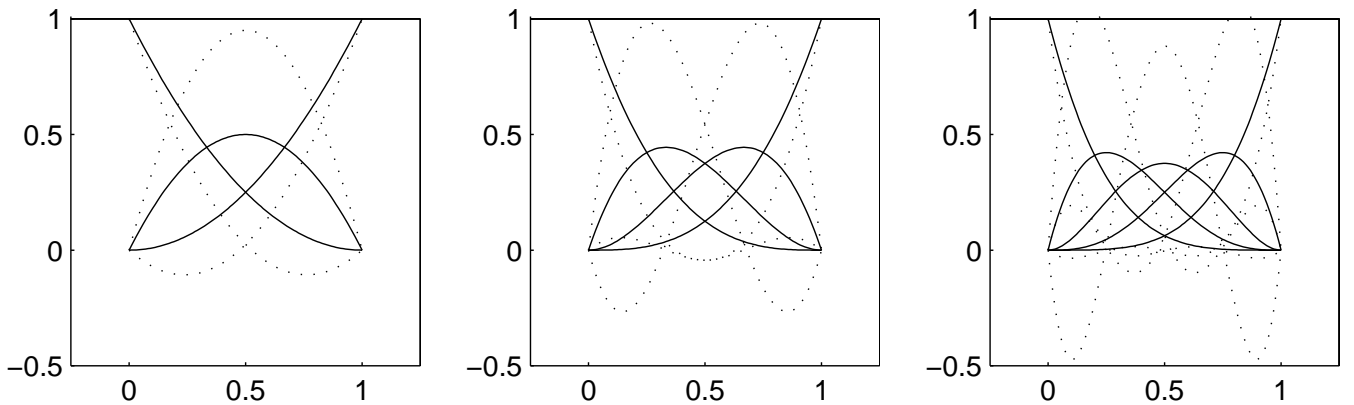


FIGURE 6.1. Dual Bases D_k^m of degree $m = 2, 3$ and 4 , for $k = 1$ (solid) and 10 (dotted), respectively. Note that $k = 1$ is the Bernstein basis, and $k = 10$ is close to the Lagrange basis.

7. RATE OF CONVERGENCE OF SYMMETRIC CONFIGURATION TO LAGRANGE INTERPOLATION

In this section we determine the rate of convergence for this symmetric configuration to Lagrange interpolation, which moreover provides an alternate proof of convergence to Lagrange interpolation. Recall that $D_k^m = B^m A_k$ is the basis for \mathbb{S}_m that is dual to $\Lambda^{mk}(s^k)$ with $s_i^k = ik$ for $i = 0 : m$, and $A_k = E(s^k, :)^{-1}$ with $E = (\Lambda^{mk})^T B^m$. Let L^m be the Lagrange basis for \mathbb{S}_m dual to point evaluation at $\frac{i}{m}$ for $i = 0 : m$, and let A be the matrix such that $L^m = B^m A$. Then, we have the following:

Lemma 7.1. *For h small:*

- (1) $(\alpha - \ell h) \cdots (\alpha - nh) = \alpha^{n-\ell+1} - \frac{1}{2} \alpha^{n-\ell} (n + \ell)(n + 1 - \ell)h + O(h^2)$.
- (2) $\frac{a - bh + O(h^2)}{c - dh + O(h^2)} = \frac{a}{c} + \frac{ad - bc}{c^2} h + O(h^2)$.

Proof. Part (2) follows by a Maclaurin's expansion. For (1), there are $n - \ell + 1$ terms. On expanding in powers of h , we have

$$\begin{aligned} (\alpha - \ell h) \cdots (\alpha - nh) &= \alpha^{n-\ell+1} - \alpha^{n-\ell} (\ell + \cdots + n)h + O(h^2) \\ &= \alpha^{n-\ell+1} - \alpha^{n-\ell} \left[\binom{n+1}{2} - \binom{\ell}{2} \right] h + O(h^2) \\ &= \alpha^{n-\ell+1} - \alpha^{n-\ell} \frac{(n+1)n - \ell(\ell-1)}{2} h + O(h^2) \\ &= \alpha^{n-\ell+1} - \frac{1}{2} \alpha^{n-\ell} (n + \ell)(n + 1 - \ell)h + O(h^2). \end{aligned}$$

□

Lemma 7.2. *For the symmetric configuration, $A^{-1}(i, j) - A_k^{-1}(i, j) = 0$ if $i = 0$ or m . Otherwise,*

$$A^{-1}(i, j) - A_k^{-1}(i, j) = C_{ij} \frac{1}{k} + O\left(\frac{1}{k^2}\right)$$

as $k \rightarrow \infty$, with

$$C_{ij} = \frac{1}{2} B_j^m \left(\frac{i}{m} \right) \left[\underbrace{\frac{(j-1)j(m-i)}{im}}_{\text{if } j > 0} + \underbrace{\frac{(m-j)(2mj - im + i - ij)}{m(m-i)}}_{\text{if } j < m} \right].$$

Proof. The Lagrange basis is dual to the point evaluation map $\Delta := [\delta_{\frac{i}{m}} : i = 0 : m]$, giving $I = \Delta^T L^m = \Delta^T B^m A$. Therefore,

$$A^{-1} = \Delta^T B^m = \left[B_j^m \left(\frac{i}{m} \right) \right].$$

We also have $A_k^{-1} = E_k(s^k, :)$ with E the degree elevation matrix. Hence,

$$A^{-1}(i, j) - A_k^{-1}(i, j) = B_j^m \left(\frac{i}{m} \right) - E(i, k, j) = B_j^m \left(\frac{i}{m} \right) - \frac{\binom{mk-m}{ik-j} \binom{m}{j}}{\binom{mk}{ik}}.$$

It is easily checked that this vanishes for $i = 0$ or $i = m$ (provided $k > 0$). Hence, we only need to consider $0 < i < m$. In this case, $ik > j$ and $mk - m \geq ik - j$ for $k \geq m$, which, since $k \rightarrow \infty$, we can assume as well. Factoring out $\binom{m}{j}$, we can rewrite the second term as

$$\begin{aligned} \frac{1}{\binom{m}{j}} E(ik, j) &= \frac{\binom{mk-m}{ik-j}}{\binom{mk}{ik}} = \frac{(ik)!}{(ik-j)!} \frac{(mk-m)!}{(mk)!} \frac{(mk-ik)!}{(mk-ik-m+j)!} \\ &= \frac{(ik) \cdots (ik-j+1)}{(mk) \cdots (mk-j+1)} \cdot \frac{(mk-ik) \cdots (mk-ik-m+j+1)}{(mk-j) \cdots (mk-m+1)} \\ &= \underbrace{\frac{i \cdots (i - \frac{j-1}{k})}{m \cdots (m - \frac{j-1}{k})}}_{\text{appears if } j > 0} \cdot \underbrace{\frac{(m-i) \cdots (m-i - \frac{m-j-1}{k})}{(m - \frac{j}{k}) \cdots (m - \frac{m-1}{k})}}_{\text{appears if } j < m}. \end{aligned}$$

By Lemma 7.1, when $j > 0$, we have

$$\begin{aligned} \frac{i \cdots (i - \frac{j-1}{k})}{m \cdots (m - \frac{j-1}{k})} &= \frac{i^j - \frac{1}{2}i^{j-1}(j-1)j\frac{1}{k} + O(\frac{1}{k^2})}{m^j - \frac{1}{2}m^{j-1}(j-1)j\frac{1}{k} + O(\frac{1}{k^2})} \\ &= \frac{i^j}{m^j} + \frac{\frac{1}{2}i^j m^{j-1}(j-1)j - \frac{1}{2}m^j i^{j-1}(j-1)j \frac{1}{k}}{(m^j)^2} + O(\frac{1}{k^2}) \\ &= \left(\frac{i}{m}\right)^j - \frac{1}{2} \left(\frac{i}{m}\right)^j \frac{(j-1)j(m-i)}{im} \frac{1}{k} + O(\frac{1}{k^2}) \\ &= \left(\frac{i}{m}\right)^j \left[1 - \frac{1}{2} \frac{(j-1)j(m-i)}{im} \frac{1}{k}\right] + O(\frac{1}{k^2}). \end{aligned}$$

Likewise, when $j < m$,

$$\begin{aligned} \frac{(m-i) \cdots (m-i - \frac{m-j-1}{k})}{(m - \frac{j}{k}) \cdots (m - \frac{m-1}{k})} &= \frac{(m-i)^{m-j} - \frac{1}{2}(m-i)^{m-j-1}(m-j-1)(m-j)\frac{1}{k} + O(\frac{1}{k^2})}{m^{m-j} - \frac{1}{2}m^{m-j-1}(m+j-1)(m-j)\frac{1}{k} + O(\frac{1}{k^2})} \\ &= \left(\frac{m-i}{m}\right)^{m-j} + \frac{\frac{1}{2}(m-i)^{m-j} m^{m-j-1}(m+j-1)(m-j) - \frac{1}{2}m^{m-j}(m-i)^{m-j-1}(m-j-1)(m-j)\frac{1}{k}}{m^{2(m-j)}} + O(\frac{1}{k^2}) \\ &= \left(1 - \frac{i}{m}\right)^{m-j} + \frac{1}{2} \left[\left(\frac{m-i}{m}\right)^{m-j} \frac{(m+j-1)(m-j)}{m} - \left(\frac{m-i}{m}\right)^{m-j} \frac{(m-j)(m-j-1)}{m-i} \right] \frac{1}{k} + O(\frac{1}{k^2}) \\ &= \left(1 - \frac{i}{m}\right)^{m-j} + \frac{1}{2} \left(\frac{m-i}{m}\right)^{m-j} (m-j) \left[\frac{m+j-1}{m} - \frac{m-j-1}{m-i} \right] \frac{1}{k} + O(\frac{1}{k^2}) \\ &= \left(1 - \frac{i}{m}\right)^{m-j} \left[1 - \frac{1}{2} \frac{(m-j)(2mj - im + i - ij)}{m(m-i)} \frac{1}{k}\right] + O(\frac{1}{k^2}). \end{aligned}$$

Multiplying these two terms together and by $\binom{m}{j}$ gives us, when $0 < j < m$,

$$\begin{aligned} E(ik, j) &= \binom{m}{j} \frac{i \cdots (i - \frac{j-1}{k})}{m \cdots (m - \frac{j-1}{k})} \cdot \frac{(m-i) \cdots (m-i - \frac{m-j-1}{k})}{(m - \frac{j}{k}) \cdots (m - \frac{m-1}{k})} \\ &= B_j^m \left(\frac{i}{m}\right) \left[1 - \frac{1}{2} \frac{(j-1)j(m-i)}{im} \frac{1}{k}\right] \left[1 - \frac{1}{2} \frac{(m-j)(2mj - im + i - ij)}{m(m-i)} \frac{1}{k}\right] + O(\frac{1}{k^2}) \\ &= B_j^m \left(\frac{i}{m}\right) - \frac{1}{2} B_j^m \left(\frac{i}{m}\right) \left[\frac{(j-1)j(m-i)}{im} + \frac{(m-j)(2mj - im + i - ij)}{m(m-i)} \right] \frac{1}{k} + O(\frac{1}{k^2}). \end{aligned}$$

For $j = 0$,

$$E(ik, 0) = B_0^m\left(\frac{i}{m}\right) + \frac{1}{2}B_0^m\left(\frac{i}{m}\right)\frac{(m-1)i}{m-i}\frac{1}{k} + O\left(\frac{1}{k^2}\right).$$

For $j = m$,

$$E(ik, m) = B_m^m\left(\frac{i}{m}\right) - \frac{1}{2}B_m^m\left(\frac{i}{m}\right)\frac{(m-1)(m-i)}{i}\frac{1}{k} + O\left(\frac{1}{k^2}\right).$$

Putting this all together, we get:

$$A^{-1}(i, j) - A_k^{-1}(i, j) = C_{ij}\frac{1}{k} + O\left(\frac{1}{k^2}\right)$$

with

$$C_{ij} = \frac{1}{2}B_j^m\left(\frac{i}{m}\right)\left[\underbrace{\frac{(j-1)j(m-i)}{im}}_{\text{if } j>0} + \underbrace{\frac{(m-j)(2mj-im+i-ij)}{m(m-i)}}_{\text{if } j<m}\right].$$

□

Theorem 7.3. *Let $p = L^m\alpha$ and $p_k = D^{m,k}\alpha$ with $L^m = B^m A$ the Lagrange basis and $D^{m,k} = B^m A_k$ with $A_k = E(s_k, \cdot)^{-1}$ the dual basis for the symmetric configuration given in the previous section. Then,*

$$\|p - p_k\|_{[0,1]} = \left(\|A\|_\infty^2 \|C\|_\infty \frac{1}{k} + O\left(\frac{1}{k^2}\right)\right) \|\alpha\|,$$

and

$$\|L_i(t) - D_i^{m,k}\|_{[0,1]} = \|A\|_\infty^2 \|C\|_\infty \frac{1}{k} + O\left(\frac{1}{k^2}\right).$$

Proof. From $A_k - A = A(A^{-1} - A_k^{-1})A_k$, we get

$$\|A_k - A\| \leq \|A\| \|A^{-1} - A_k^{-1}\| \|A_k\|.$$

Since, as we proved in the previous section, that $D^{m,k}$ converges to the Lagrange basis L^m , we know that $A_k \rightarrow A$. Hence, $\limsup_k \|A_k\|_\infty \leq \|A\|_\infty$. We also know that

$$\|A^{-1} - A_k^{-1}\|_\infty = \|C\|_\infty \frac{1}{k} + O\left(\frac{1}{k^2}\right).$$

Therefore,

$$\|A_k - A\|_\infty = \|A\|_\infty^2 \|C\|_\infty \frac{1}{k} + O\left(\frac{1}{k^2}\right).$$

Now, let $p = L^m\alpha = B^m A\alpha$ and $p_k = D^{m,k}\alpha = B^m A_k\alpha$ for $\alpha \in \mathbb{R}^{m+1}$. Then,

$$\begin{aligned} \|p - p_k\|_{[0,1]} &= \|B^m(A - A_k)\alpha\|_{[0,1]} \\ &= \max_{t \in [0,1]} \|B^m(t)(A - A_k)\alpha\| \\ &\leq \max_{t \in [0,1]} \|B^m(t)\|_\infty \|A - A_k\|_\infty \|\alpha\|_\infty \\ &= \|A - A_k\|_\infty \|\alpha\|_\infty \\ &= \left(\|A\|_\infty^2 \|C\|_\infty \frac{1}{k} + O\left(\frac{1}{k^2}\right)\right) \|\alpha\|_\infty. \end{aligned}$$

In particular, for $\alpha = e_i$ the standard unit vector with 1 in the i -th slot, we get

$$\|L_i(t) - D_i^{m,k}\|_{[0,1]} = \|A\|_\infty^2 \|C\|_\infty \frac{1}{k} + O\left(\frac{1}{k^2}\right).$$

□

8. BASIS TRANSFORMATIONS FOR SYMMETRIC CLASS

Recall that for the symmetric configuration introduced in the previous section the dual basis is represented in terms of the Bernstein basis $D^{m,k} = B^m A_{m,k}$ for some $m \times m$ matrices $A_{m,k}$. It seems rather challenging to explicitly characterize all these transformation matrices for arbitrary m and k , however, for the convenience of the reader we'll list out the first several here. To make things more compact, we define the following notation: $km := k - m$, $nk m := nk - m$, $pk^2 nk m = pk^2 - nk + m$, and $qk^3 pk^2 nk m = qk^3 - pk^2 + nk - m$.

$$A_{2,k} = \frac{1}{2!k} \begin{bmatrix} 2k & 0 & 0 \\ -(k1) & 2(2k1) & -(k1) \\ 0 & 0 & 2k \end{bmatrix}.$$

$$A_{3,k} = \frac{1}{2 \cdot 3!k^2} \begin{bmatrix} 12k^2 & 0 & 0 & 0 \\ -2(k1)(5k1) & 6(2k1)(3k1) & -6(k1)(3k1) & 2(k1)(2k1) \\ 2(k1)(2k1) & -6(k1)(3k1) & 6(2k1)(3k1) & -2(k1)(5k1) \\ 0 & 0 & 0 & 12k^2 \end{bmatrix}.$$

$$A_{4,k} = \frac{1}{3 \cdot 4!k^3} \begin{bmatrix} 72k^3 & 0 & 0 & 0 & 0 \\ -3(k1)(26k^2 9k1) & 12(2k1)(3k1)(4k1) & -18(k1)(3k1)(4k1) & 12(k1)(2k1)(4k1) & -3(k1)(2k1)(3k1) \\ 4(k1)(13k^2 12k2) & -32(k1)(2k1)(4k1) & 24(4k1)(5k^2 6k2) & -32(k1)(2k1)(4k1) & 4(k1)(13k^2 12k2) \\ -3(k1)(2k1)(3k1) & 12(k1)(2k1)(4k1) & -18(k1)(3k1)(4k1) & 12(2k1)(3k1)(4k1) & -3(k1)(26k^2 9k1) \\ 0 & 0 & 0 & 0 & 72k^3 \end{bmatrix}.$$

$$A_{5,k} = \frac{1}{4 \cdot 5!k^4} \begin{bmatrix} 480k^4 & 0 & 0 & 0 & 0 \\ -4(k1)(7k1)(22k^2 7k1) & 20(2k1)(3k1)(4k1)(5k1) & -40(k1)(3k1)(4k1)(5k1) & 40(k1)(2k1)(4k1)(5k1) & 0 \\ 2(k1)(269k^3 331k^2 109k11) & -10(k1)(2k1)(5k1)(29k11) & 20(5k1)(59k^3 101k^2 59k11) & -20(k1)(2k1)(5k1)(23k11) & 4(k1)(2k1)(3k1)(4k1) \\ -2(k1)(2k1)(77k^2 72k11) & 10(k1)(5k1)(37k^2 42k11) & -20(k1)(2k1)(5k1)(23k11) & 20(5k1)(59k^3 101k^2 59k11) & -2(k1)(2k1)(77k^2 72k11) \\ 4(k1)(2k1)(3k1)(4k1) & -20(k1)(2k1)(3k1)(5k1) & 40(k1)(2k1)(4k1)(5k1) & -40(k1)(3k1)(4k1)(5k1) & 2(k1)(269k^3 331k^2 109k11) \\ 0 & 0 & 20(2k1)(3k1)(4k1)(5k1) & -4(k1)(7k1)(22k^2 7k1) & 0 \\ 0 & 0 & 0 & 0 & 480k^4 \end{bmatrix}.$$

In particular, for $m = 2$ and $k = 2 : 5$:

$$\frac{1}{4} \begin{bmatrix} 4 & 0 & 0 \\ -1 & 6 & -1 \\ 0 & 0 & 4 \end{bmatrix}, \frac{1}{6} \begin{bmatrix} 6 & 0 & 0 \\ -2 & 10 & -2 \\ 0 & 0 & 6 \end{bmatrix}, \frac{1}{8} \begin{bmatrix} 8 & 0 & 0 \\ -3 & 14 & -3 \\ 0 & 0 & 8 \end{bmatrix}, \frac{1}{10} \begin{bmatrix} 10 & 0 & 0 \\ -4 & 18 & -4 \\ 0 & 0 & 10 \end{bmatrix}.$$

For $m = 3$ and $k = 2 : 5$:

$$\frac{1}{48} \begin{bmatrix} 48 & 0 & 0 & 0 \\ -18 & 90 & -30 & 6 \\ 6 & -30 & 90 & -18 \\ 0 & 0 & 0 & 48 \end{bmatrix}, \frac{1}{108} \begin{bmatrix} 108 & 0 & 0 & 0 \\ -56 & 240 & -96 & 20 \\ 20 & -96 & 240 & -56 \\ 0 & 0 & 0 & 108 \end{bmatrix}, \frac{1}{192} \begin{bmatrix} 192 & 0 & 0 & 0 \\ -114 & 462 & -198 & 42 \\ 42 & -198 & 462 & -114 \\ 0 & 0 & 0 & 192 \end{bmatrix}, \frac{1}{300} \begin{bmatrix} 300 & 0 & 0 & 0 \\ -192 & 756 & -336 & 72 \\ 72 & -336 & 756 & -192 \\ 0 & 0 & 0 & 300 \end{bmatrix}.$$

For $m = 4$ and $k = 2 : 4$:

$$\frac{1}{576} \begin{bmatrix} 576 & 0 & 0 & 0 & 0 \\ -261 & 1260 & -630 & 252 & -45 \\ 120 & -672 & 1680 & -672 & 120 \\ -45 & 252 & -630 & 1260 & -261 \\ 0 & 0 & 0 & 0 & 576 \end{bmatrix}, \frac{1}{1944} \begin{bmatrix} 1944 & 0 & 0 & 0 & 0 \\ -1248 & 5280 & -3168 & 1320 & -240 \\ 664 & -3520 & 7656 & -3520 & 644 \\ -240 & 1320 & -3168 & 5280 & -1248 \\ 0 & 0 & 0 & 0 & 1944 \end{bmatrix},$$

$$\frac{1}{4608} \begin{bmatrix} 4608 & 0 & 0 & 0 & 0 \\ -3429 & 13860 & -8910 & 3780 & -693 \\ 1944 & -10080 & 20880 & -10080 & 1944 \\ -693 & 3780 & -8910 & 13860 & -3429 \\ 0 & 0 & 0 & 0 & 4608 \end{bmatrix}.$$

For $m = 5$ and $k = 2 : 3$:

$$\frac{1}{38880} \begin{bmatrix} 38880 & 0 & 0 & 0 & 0 & 0 \\ -28480 & 123200 & -98560 & 61600 & -22400 & 3520 \\ 18400 & -106400 & 238000 & -162400 & 61040 & -9760 \\ -9760 & 61040 & -162400 & 238000 & -106400 & 18400 \\ 3520 & -22400 & 61600 & -98560 & 123200 & -28480 \\ 0 & 0 & 0 & 0 & 0 & 38880 \end{bmatrix},$$

$$\frac{1}{122880} \begin{bmatrix} 122880 & 0 & 0 & 0 & 0 & 0 \\ -105300 & 438900 & -376200 & 239400 & -87780 & 13860 \\ 74070 & -418950 & 906300 & -646380 & 247950 & -40110 \\ -40110 & 247950 & -646380 & 906300 & -418950 & 74070 \\ 13860 & -87780 & 239400 & -376200 & 438900 & -105300 \\ 0 & 0 & 0 & 0 & 0 & 122880 \end{bmatrix}$$

9. QUASI-INTERPOLATION

In this section we derive some approximation results for our dual basis functions, similar to what is done in [7] in the multivariate setting. To do so, we will redefine the Bernstein basis and dual functionals over a general interval.

- Let B^n with

$$B_i^n = \binom{n}{i} \left(\frac{b - \cdot}{b - a} \right)^{n-i} \left(\frac{\cdot - a}{b - a} \right)^i$$

be the Bernstein basis over $[a, b]$, with dual functionals

$$\lambda_k^n = \sum_{j=0}^k \frac{\binom{k}{j} (b - a)^j}{\binom{n}{j} j!} \delta_a D^j.$$

Hence, $\Lambda^{nT} B^n = I$.

- Let D^m be the basis for \mathcal{S}^m dual to $\Lambda^n(s)$ for some selection s . That is, $\Lambda^{nT}(s)D^m = I$. Then, $D^m = B^m A$ with $A = E(s, \cdot)^{-1}$ where $B^n = B^m E$.
- Let L^m be the Lagrange basis at the points $\xi^m = [a + \frac{i}{m}(b-a) : i = 0 : m]$ with dual map $\Delta^m = [\delta_{\xi_0^m}, \dots, \delta_{\xi_m^m}]$. Hence, $\Delta^{mT}L^m = I$. Then, for $B^m = L^m M_m$ with $M_m = \Delta^{mT}B^m$, is the basis transformation. Let $\tilde{\Lambda}^m = \Delta^m M_m^{-T}$.

The purpose of the map $\tilde{\Lambda}^m$ is two-fold. First, it is dual to B^m , as we show next. Second, it does not involve any derivative evaluations, which makes it suitable for the approximation of functions in $C[a, b]$.

Lemma 9.1. $\tilde{\Lambda}^n$ is dual to B^n .

Proof.

$$\tilde{\Lambda}^{nT}B^n = (\Delta^n M^{-T})^T B^n = M^{-1} \Delta^{nT} B^n = M^{-1} M = I.$$

□

Theorem 9.2. (Stability) Let $p = D^m \alpha \in \mathcal{S}^m$. Then,

$$\frac{1}{\|M_m^{-1}\|_\infty} \|\alpha\|_\infty \leq \|p\|_{[a,b]} \leq \|A\|_\infty \|\alpha\|_\infty.$$

Proof. For the upper bound:

$$\begin{aligned} \|p\|_{[0,1]} &= \|D^m \alpha\|_{[0,1]} = \|B^m A \alpha\|_{[0,1]} = \left\| \sum_{i=0}^m (A\alpha)_i B_i^m \right\|_{[0,1]} \\ &\leq \|A\alpha\|_\infty \left\| \sum_{i=0}^m B_i^m \right\|_{[0,1]} = \|A\alpha\|_\infty \leq \|A\|_\infty \|\alpha\|_\infty. \end{aligned}$$

For the lower bound, note that

$$\Delta^{mT} p = \Delta^{mT} D^m \alpha = \Delta^{mT} B^m A \alpha = M_m A \alpha,$$

and so

$$\alpha = A^{-1} M_m^{-1} \Delta^{mT} p.$$

Then,

$$\begin{aligned} \|\alpha\|_\infty &\leq \|A^{-1}\|_\infty \|M_m^{-1}\|_\infty \|\Delta^{mT} p\|_\infty \\ &\leq \|A^{-1}\|_\infty \|M_m^{-1}\|_\infty \|p\|_{[0,1]}. \end{aligned}$$

Since $A^{-1} = E(s, \cdot)$ is row-affine, $\|A^{-1}\|_\infty = 1$, and so we get

$$\frac{1}{\|M_m^{-1}\|_\infty} \|\alpha\|_\infty \leq \|p\|_{[0,1]}.$$

□

For this we recall from Proposition 3.6 that the dual basis is unaffected by which map is used. Hence, we restate this fact in the following lemma:

Lemma 9.3. *Suppose that data maps Λ^n and $\tilde{\Lambda}^n$ are both dual to the basis B^n for \mathbb{S}^n . Then, these data maps are equivalent on \mathbb{S}^n , and dual bases D^m of \mathbb{S}^m for $m < n$ are identical for both maps.*

Lemma 9.4. *For $f \in C([a, b])$,*

$$|\tilde{\lambda}_j^n f| \leq \|f\|_{[a,b]} \|M^{-1}\|_\infty.$$

Proof. Let $C := M^{-T}$. Then, $\tilde{\Lambda}^m = \Delta C$, and so

$$\begin{aligned} |\tilde{\lambda}_j^n f| &= |\Delta C(:, j) f| = \left| \sum_{i=0}^n C(i, j) \delta_{\frac{i}{n}} f \right| = \left| \sum_{i=0}^n C(i, j) f\left(\frac{i}{n}\right) \right| \\ &\leq \|f\|_{[a,b]} \sum_{i=0}^n |C(i, j)| \leq \|f\|_{[a,b]} \|M^{-T}\|_1 = \|f\|_{[a,b]} \|M^{-1}\|_\infty. \end{aligned}$$

□

For any selection s , let

$$Q_s : C([a, b]) \rightarrow \mathbb{R} : f \mapsto D^m \tilde{\Lambda}^n(s)^T f.$$

Then, we have following:

Lemma 9.5. *Q_s is a linear projector of $C([a, b])$ onto \mathbb{S}^m .*

Proof. Linearity is immediate from

$$Q_s(\alpha f + \beta g) = D^m \tilde{\Lambda}^n(s)^T (\alpha f + \beta g) = \alpha D^m \tilde{\Lambda}^n(s)^T f + \beta D^m \tilde{\Lambda}^n(s)^T g = \alpha Q_s f + \beta Q_s g,$$

and idempotency follows from

$$Q_s^2 = (D^m \tilde{\Lambda}^n(s)^T)^2 = D^m \left(\tilde{\Lambda}^n(s)^T D^m \right) \tilde{\Lambda}^n(s)^T = D^m \tilde{\Lambda}^n(s)^T = Q_s.$$

since $\tilde{\Lambda}^n(s)^T D^m = I$. Therefore, Q_s is a linear projector. □

Theorem 9.6. *Let $f \in C[a, b]$. Then,*

- $\|Q_s f\|_{[a,b]} \leq \|A\|_\infty \|M_m^{-1}\|_\infty \|f\|_{[a,b]}$,
- $\|f - Q_s f\|_{[a,b]} \leq (1 + \|Q_s\|) d(f, \mathbb{S}_m)_{[a,b]}$.

Proof. For $t \in [a, b]$,

$$\begin{aligned} |Q_s f(t)| &= |(D^m \tilde{\Lambda}^n(s)^T f)(t)| = |(B^m A \tilde{\Lambda}^n(s)^T f)(t)| \\ &\leq \|A\|_\infty \|\Lambda^n(s)^T f\|_\infty \sum_{i=0}^m B_i^m(t) = \|A\|_\infty \|\Lambda^n(s)^T f\|_\infty \\ &\leq \|A\|_\infty \|M^{-1}\|_\infty \|f\|_{[a,b]}. \end{aligned}$$

This establishes the first result. For the second result, take an arbitrary $p \in \mathcal{S}_m$. Then,

$$\begin{aligned} \|f - Q_s f\|_{[a,b]} &\leq \|f - p\|_{[a,b]} + \underbrace{\|p - Q_s p\|_{[a,b]}}_0 + \|Q_s(p - f)\|_{[a,b]} \\ &\leq (1 + \|Q_s\|_{[a,b]})\|f - p\|_{[a,b]} \\ &\leq (1 + \|Q_s\|_{[a,b]})d(f, \mathcal{S}_m)_{[a,b]}. \end{aligned}$$

□

10. BERNSTEIN-LIKE OPERATOR

Let

$$D_m f := D^m \Delta^{nT}(s) f = \sum_{i=0}^m f(\xi_{s(i)}^n) D_i^m$$

be a Bernstein-like operator for our Dual functions, with $\xi_j^n := a + \frac{j}{n}(b - a)$.

Let

$$\|f\|_{k,[a,b]} := \max\{|f^{(k)}(x)| : x \in [a, b]\},$$

with $\|f\|_{[a,b]} := \|f\|_{0,[a,b]}$, and

$$\omega(f, h) := \max\{|f(x) - f(y)| : x, y \in [a, b], |x - y| \leq h\},$$

the uniform modulus of continuity relative to the interval $[a, b]$. Then, we have the following:

Theorem 10.1.

$$\|f - D_m f\|_{[a,b]} \leq \begin{cases} \|A\|_\infty \omega(f, b - a), & f \in C([a, b]); \\ (b - a) \|A\|_\infty \|f'\|_{[a,b]}, & f \in C^1([a, b]); \\ \frac{1}{2}(b - a)^2 \|A\|_\infty \|f''\|_{[a,b]}, & f \in C^2([a, b]). \end{cases}$$

Proof. By affineness of the basis $D^m = B^m A$ and convexity of B^m for all $x \in [a, b]$,

$$\begin{aligned} |f(x) - D_m f(x)| &= \left| \sum_{i=0}^m (f(x) - f(\xi_{s(i)}^n)) D_i^m(x) \right| \\ &\leq \omega(f, b - a) \sum_{i=0}^m |D_i^m(x)| \\ &= \omega(f, b - a) \left| \sum_{j=0}^m B_j^m(x) \sum_{i=0}^m A(j, i) \right| \\ &\leq \omega(f, b - a) \|A\|_\infty \sum_{j=0}^m |B_j^m(x)| \\ &= \|A\|_\infty \omega(f, b - a). \end{aligned}$$

This gives the first case. The second case follows from the first case and the estimate

$$\begin{aligned}
\omega(f, h) &= \max_{|x-y| \leq h} |f(x) - f(y)| \\
&= \max_{|x-y| \leq h} \frac{|f(x) - f(y)|}{|x-y|} |x-y| \\
&\leq \max_{|x-y| \leq h} \frac{|f(x) - f(y)|}{|x-y|} h \\
&\leq h \|f'\|_{[a,b]}.
\end{aligned}$$

Then, for some η between $\xi_{s(i)}^n$ and x ,

$$\begin{aligned}
|f(x) - D_m f(x)| &= |f(x) - \sum_{i=0}^m f(\xi_{s(i)}^n) D_i^m| \\
&= |f(x) - \sum_{i=0}^m \left[f(x) + f'(x)(\xi_{s(i)}^n - x) + \frac{1}{2} f''(\eta)(\xi_{s(i)}^n - x)^2 \right] D_i^m| \\
&= |f(x) - \sum_{i=0}^m f(x) D_i^m + \sum_{i=0}^m f'(x)(\xi_{s(i)}^n - x) D_i^m + \frac{1}{2} \sum_{i=0}^m f''(\eta)(\xi_{s(i)}^n - x)^2 D_i^m| \\
&= \left| \sum_{i=0}^m f'(x)(\xi_{s(i)}^n - x) D_i^m + \frac{1}{2} \sum_{i=0}^m f''(\eta)(\xi_{s(i)}^n - x)^2 D_i^m \right| \\
&= |f'(x) \sum_{i=0}^m \xi_{s(i)}^n D_i^m - f'(x)x + \frac{1}{2} \sum_{i=0}^m f''(\eta)(\xi_{s(i)}^n - x)^2 D_i^m|
\end{aligned}$$

By Theorem 4.3 (transformed to the interval $[a, b]$), $\sum_{i=0}^m \xi_{s(i)}^n D_i^m = x$, and so

$$\begin{aligned}
|f(x) - D_m f(x)| &= |f'(x)x - f'(x)x + \frac{1}{2} \sum_{i=0}^m f''(\eta)(\xi_{s(i)}^n - x)^2 D_i^m| \\
&= \left| \frac{1}{2} \sum_{i=0}^m f''(\eta)(\xi_{s(i)}^n - x)^2 D_i^m \right| \\
&\leq \frac{1}{2} \|f''\|_{[a,b]} (b-a)^2 \sum_{i=0}^m |D_i^m(x)| \\
&\leq \frac{1}{2} \|f''\|_{[a,b]} (b-a)^2 \|A\|.
\end{aligned}$$

□

REFERENCES

1. G. Farin, *Curves and Surfaces for Computer Aided Geometric Design*, 1988.
2. R. Goldman, Dual Polynomial Bases, *JAT* 79(3), 1994, 311–346.
3. S. Kersey, Invertibility of submatrices of Pascal's matrix, arXiv:1303.6159 (2013).

4. S. Kersey, Dual basis functions in subspaces of inner product spaces, AMC 219 (2013) 10012–10024.
5. Y. Katznelson and Y. Katznelson, A (Terse) Introduction to Linear Algebra, AMS (Student Mathematical Library) (44), (2008).
6. H. Prautzsch, W. Boehm and M. Paluszny, Bézier and B-spline techniques, Springer (New York) (2002).
7. L. Schumaker, M. Lai, Spline Functions on Triangulations, Encyclopedia of Mathematics and its Applications, Volume 110, Cambridge University Press (Cambridge, UK), (2007).

Current address: Department of Mathematical Science, Georgia Southern University, Statesboro, GA 30460-8093

E-mail address: `skersey@georgiasouthern.edu`