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Bohr density of simple linear group orbits

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Abstract. We show that any nonzero orbit under a noncompact, simple, irreducible linear group is dense in the Bohr compactification of the ambient space.

1. *Introduction*

Let V be a locally compact abelian group, V^* its Pontrjagin dual and bV its Bohr compactification, i.e. bV is the dual of the discretized group V^*. On identifying V with its double dual we have a dense embedding $V \hookrightarrow bV$, viz.

$$\{\text{continuous characters of } V^*\} \hookrightarrow \{\text{all characters of } V^*\}.$$

The relative topology of V in bV is known as the *Bohr topology* of V. Among its many intriguing properties (surveyed in [**G07**]) is the observation due to Katznelson [**K73a**; **G79**, §7.6] that very "thin" subsets of V can be Bohr dense in very large ones.

While Katznelson was concerned with the case $V = \mathbf{Z}$ (the integers), we shall illustrate this phenomenon in the setting where V is the additive group of a real vector space, and the subsets of interest are the orbits of a Lie group acting linearly on V. Indeed our aim is to establish the following result, which was conjectured in [**Z96**, p. 45]:

Theorem 1. *Let G be a noncompact, simple real Lie group and V a nontrivial, irreducible, finite-dimensional real G-module. Then every nonzero G-orbit in V is dense in bV.*

We prove this in §3 on the basis of four lemmas prepared in §2. Before that, let us record a similar property of *nilpotent* groups. In that case, orbits typically lie in proper affine subspaces, so we can't hope for Bohr density in the whole space; but we have:

Theorem 2. *Let G be a connected nilpotent Lie group and V a finite-dimensional G-module of unipotent type. Then every G-orbit in V is Bohr dense in its affine hull.*

Proof. Recall that *unipotent type* means that the Lie algebra $\mathfrak{g}$ of G acts by nilpotent operators. So $Z \mapsto \exp(Z)v$ is a polynomial map of $\mathfrak{g}$ onto the orbit of $v \in V$, and the claim follows immediately from [**Z93**, Theorem]. □

2. *Four lemmas*

Our first lemma gives several characterizations of Bohr density — each of which can also be regarded as providing a corollary of Theorem 1.

Lemma 1. *Let O be a subset of the locally compact abelian group V. Then the following are equivalent*:

(1) *O is dense in bV*;

(2) *$\alpha(O)$ is dense in $\alpha(V)$ whenever α is a continuous morphism from V to a compact topological group*;

(3) *Every almost periodic function on V is determined by its restriction to O*;

(4) *Haar measure η on bV is the weak$*$ limit of probability measures μ_T concentrated on O.*

Proof. (1) ⇔ (2): Clearly (2) implies (1) as the special case where α is the natural inclusion $\iota : V \hookrightarrow bV$. Conversely, suppose (1) holds and $\alpha : V \to X$ is a continuous morphism to a compact group. By the universal property of bV [D82, 16.1.1], $\alpha = \beta \circ \iota$ for a continuous morphism $\beta : bV \to X$. Now continuity of β implies $\beta(\overline{\iota(O)}) \subset \overline{\beta(\iota(O))}$, which is to say that $\beta(bV) \subset \overline{\alpha(O)}$ and hence $\alpha(V) \subset \overline{\alpha(O)}$, as claimed.

(1) ⇔ (3): Recall that a function on V is *almost periodic* iff it is the pull-back of a continuous $f : bV \to \mathbf{C}$ by the inclusion $V \hookrightarrow bV$. If two such functions coincide on O and O is dense in bV, then clearly they coincide everywhere. Conversely, suppose that O is not dense in bV. Then by complete regularity [H63, 8.4] there is a nonzero continuous $f : bV \to \mathbf{C}$ which is zero on the closure of O in bV. Now clearly this f is not determined by its restriction to O.

(1) ⇔ (4) ([K73a]): Suppose that η is the weak$*$ limit of probability measures μ_T concentrated on O. So we have $\mu_T(f) \to \eta(f)$ for every continuous f, and the complement of O in bV is μ_T-null [B04, Def. V.5.7.4 and Prop. IV.5.2.5]. If f vanishes on the closure of O in bV then so do all $\mu_T(|f|)$ and hence also $\eta(|f|)$, which forces f to vanish everywhere. So O is dense in bV. Conversely, suppose that O is dense in bV. We have to show that given continuous functions $f_1, \dots, f_n$ on bV and $\varepsilon > 0$, there is a probability measure μ concentrated on O such that $|\eta(f_j) - \mu(f_j)| < \varepsilon$ for all j. Writing

$$F = (f_1, \dots, f_n) \quad \text{and} \quad \eta(F) = (\eta(f_1), \dots, \eta(f_n))$$

we see that this amounts to $\|\eta(F) - \mu(F)\| < \varepsilon$, where the norm is the sup norm in $\mathbf{C}^n$. Now by [B04, Cor. V.6.1] $\eta(F)$ lies in the convex hull of $F(bV)$ (which is compact by Carathéodory's theorem [B87, 11.1.8.7]). So $\eta(F)$ is a convex combination $\sum_{i=1}^N \lambda_i F(\omega_i)$ of elements of $F(bV)$. But $F(O)$ is dense in $F(bV)$, so we can find $w_i \in O$ such that $\|F(\omega_i) - F(w_i)\| < \varepsilon$. Putting $\mu = \sum_{i=1}^N \lambda_i \delta_{w_i}$ where δ_{w_i} is Dirac measure at w_i, we obtain the desired probability measure μ. □

Remark 1. One might wonder if condition (2) is equivalent to the following *a priori* weaker but already interesting property:

(2′) O has dense image in any compact quotient group of V.

Here is an example showing that (2′) *does not* imply (2): Let $V = \mathbf{R}$ and $O = \mathbf{Z} \cup 2\pi\mathbf{Z}$. Then clearly O has dense image in every compact quotient $\mathbf{R}/a\mathbf{Z}$. On the other hand, considering the irrational winding $\alpha : \mathbf{R} \to \mathbf{T}^2$ defined by $\alpha(v) = (e^{iv}, e^{2\pi i v})$, one checks without trouble that $\overline{\alpha(O)} = \mathbf{T} \times \{1\} \cup \{1\} \times \mathbf{T}$, which is strictly smaller than $\overline{\alpha(V)} = \mathbf{T}^2$.

Remark 2. A net of probability measures μ_T converging to Haar measure on bV as in (4) has been called a *generalized summing sequence* by Blum and Eisenberg [**B74**]. They observed, among others, the following characterization.

LEMMA 2. *The following conditions are equivalent*:
(1) μ_T *is a generalized summing sequence*;
(2) *The Fourier transforms* $\hat{\mu}_T(u) = \int_{bV} \omega(u) d\mu_T(\omega)$ *converge pointwise to the characteristic function of* $\{0\} \subset V^*$.

Proof. This characteristic function is the Fourier transform of Haar measure η on bV. Thus, condition (2) says that $\mu_T(f) \to \eta(f)$ for every continuous character $f(\omega) = \omega(u)$ of bV; whereas condition (1) says that $\mu_T(f) \to \eta(f)$ holds for every continuous function f on bV. Since linear combinations of continuous characters are uniformly dense in the continuous functions on bV (Stone-Weierstrass), the two conditions imply each other. □

For our third lemma, let G be a group, V a finite-dimensional G-module, and write V^* for the dual module wherein G acts contragrediently: $\langle gu, v\rangle = \langle u, g^{-1}v\rangle$. We have

LEMMA 3. *Suppose that* V *is irreducible and* $\phi(g) = \langle u, gv\rangle$ *is a nonzero matrix coefficient of* V. *Then every other matrix coefficient* $\psi(g) = \langle x, gy\rangle$ *is a linear combination of left and right translates of* ϕ.

Proof. Irreducibility of V and (therefore) V^* ensures that u and v are cyclic, i.e. their G-orbits span V^* and V. So we can write $x = \sum_i \alpha_i g_i u$ and $y = \sum_j \beta_j g_j v$, whence $\psi(g) = \sum_{i,j} \alpha_i \beta_j \phi(g_i^{-1} g g_j)$. □

Finally, our fourth preliminary result is the famous

LEMMA 4 (VAN DER CORPUT) *Suppose that* $F : [a, b] \to \mathbf{R}$ *is differentiable, its derivative* F' *is monotone, and* $|F'| \geqslant 1$ *on* (a, b). *Then* $\left|\int_a^b e^{iF(t)} dt\right| \leqslant 3$.

Proof. See [**S93**, p. 332], or [**R05**, Lemma 3] which actually gives the sharp bound 2. □

3. *Proof of Theorem 1*

By Lemma 1, it is enough to show that Haar measure on bV is the weak$*$ limit of probability measures μ_T concentrated on the orbit under consideration; or equivalently (Lemma 2), that the Fourier transforms of the μ_T tend pointwise to the characteristic function of $\{0\} \subset V^*$. (Here we identify the Pontrjagin dual with the dual vector space or module.)

To construct such μ_T, we assume without loss of generality that the action of G on V is effective, so that we may regard $G \subset \mathrm{GL}(V)$. Let $K \subset G$ be a maximal compact subgroup, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition, $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subalgebra, $C \subset \mathfrak{a}^*$ a Weyl chamber, $P \subset \mathfrak{a}$ the dual positive cone, and H an interior point of P; thus we have that $\langle v, H\rangle$ is positive for all nonzero $v \in C$. (For all this structure see, for example, [**K73b**].)

We fix a nonzero $v \in V$, and for each positive $T \in \mathbf{R}$ we let μ_T denote the image of the product measure Haar $\times$ (Lebesgue$/T$) $\times$ Haar under the composed map

$$\begin{array}{rcccl} K \times [0,T] \times K & \longrightarrow & Gv & \longrightarrow & bV \\ (k,t,k') & \longmapsto & k\exp(tH)k'v & & \\ & & w & \longmapsto & \mathrm{e}^{\mathrm{i}\langle\cdot,w\rangle}. \end{array}$$

Here $\exp : \mathfrak{a} \to A$ is the usual matrix exponential with inverse $\log : A \to \mathfrak{a}$, and the brackets $\langle\cdot,\cdot\rangle$ denote both pairings, $\mathfrak{a}^* \times \mathfrak{a} \to \mathbf{R}$ and $V^* \times V \to \mathbf{R}$. By construction the μ_T are concentrated on the subset Gv of bV [**B04**, Cor. V.6.2.3]. There remains to show that as $T \to \infty$ we have, for every nonzero $u \in V^*$,

$$\int_{K\times K} dk\,dk'\,\frac{1}{T}\int_0^T \mathrm{e}^{\mathrm{i}\langle u,k\exp(tH)k'v\rangle}dt \to 0. \qquad (*)$$

To this end, let

$$F_{kk'}(t) = \langle u, k\exp(tH)k'v\rangle$$

denote the exponent in ($*$). We are going to show that Lemma 4 applies to almost every $F_{kk'}$. In fact, it is well known (see for example [**K73b**, Prop. 2.4 and proof of Prop. 3.4]) that $\mathfrak{a}$ acts diagonalizably (over $\mathbf{R}$) on V. Thus, letting E_ν be the projector of V onto the weight ν eigenspace of $\mathfrak{a}$, we can write

$$F_{kk'}(t) = \sum_{\nu\in\mathfrak{a}^*}\langle u, kE_\nu k'v\rangle \mathrm{e}^{\langle\nu,H\rangle t}.$$

Now we claim that there are nonzero ν such that the coefficient $f_\nu(k,k') = \langle u, kE_\nu k'v\rangle$ is not identically zero on $K \times K$. (Then f_ν, being analytic, will be nonzero *almost everywhere*.) Indeed, suppose otherwise. Then, writing any $g \in G$ in the form kak' (KAK decomposition [**K02**]), we would have

$$\langle u, gv\rangle = \sum_{\nu\in\mathfrak{a}^*}\langle u, kE_\nu k'v\rangle \mathrm{e}^{\langle\nu,\log(a)\rangle} = \langle u, kE_0k'v\rangle.$$

In particular the matrix coefficient $\langle u, gv\rangle$ would be bounded. Hence so would be all matrix coefficients, since they are linear combinations of translates of this one (Lemma 3); and this would contradict the noncompactness of $G \subset \mathrm{GL}(V)$.

So the set $N = \{\nu \in \mathfrak{a}^* : \nu \neq 0, f_\nu \neq 0\}$ is not empty. It is also Weyl group invariant, hence contains weights $\nu \in C$ for which we know $\langle\nu, H\rangle$ is positive. Therefore, maximizing $\langle\nu, H\rangle$ over N produces a positive number $\langle\nu_0, H\rangle$, in terms of which our exponent and its derivatives can be written

$$\frac{d^n}{dt^n}F_{kk'}(t) = \mathrm{e}^{\langle\nu_0,H\rangle t}\sum_{\nu\in\mathfrak{a}^*} f_\nu(k,k')\langle\nu,H\rangle^n \mathrm{e}^{\langle\nu-\nu_0,H\rangle t}$$

where $\langle\nu - \nu_0, H\rangle < 0$ in all nonzero terms except the one indexed by ν_0. (Here we assume, as we may, that H was initially chosen outside the kernels of all pairwise differences of weights of V.) From this it is clear that for almost all (k,k') there is a T_0 beyond which the first two derivatives of $F_{kk'}$ are greater than 1 in absolute value. So Lemma 4 applies and gives

$$\left|\int_{T_0}^T \mathrm{e}^{\mathrm{i}F_{kk'}(t)}dt\right| \leqslant 3 \qquad \forall\, T.$$

Therefore we have $\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{iF_{kk'}(t)}dt = 0$ for almost all (k,k'), whence the conclusion ($*$) by dominated convergence. This completes the proof.

4. *Outlook*

Theorem 1 says that the G-action on $V\setminus\{0\}$ is *minimal* [P83] in the Bohr topology. It would be interesting to determine if it is still minimal, and/or *uniquely ergodic*, on $bV\setminus\{0\}$.

It is also natural to speculate whether our theorems have a common extension to more general group representations. Here we shall content ourselves with noting two obstructions. First, Theorem 1 clearly fails for *semi*simple groups with compact factors. Secondly, Theorem 2 fails for V not of unipotent type, as one sees by observing that the orbits of $\mathbf{R}$ acting on $\mathbf{R}^2$ by $\exp\left(\begin{smallmatrix} t & 0\\ 0 & -t\end{smallmatrix}\right)$ (i.e., hyperbolas) already have non-dense images in $\mathbf{R}^2/\mathbf{Z}^2$.

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