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Local Rings of Embedding Codepth at Most 3 Have Only Trivial Semidualizing Complexes

Saeed Nasseh
Georgia Southern University, snasseh@georgiasouthern.edu

Sean Sather-Wagstaff
Clemson University

LOCAL RINGS OF EMBEDDING CODEPTH AT MOST 3 HAVE ONLY TRIVIAL SEMIDUALIZING COMPLEXES

SAEED NASSEH AND SEAN SATHER-WAGSTAFF

Abstract. We prove that a local ring R of embedding codepth at most 3 has at most two semidualizing complexes up to shift-isomorphism, namely, R itself and a dualizing R-complex if one exists.

1. Introduction

Convention. In this paper, R is a commutative noetherian ring. In this section, assume that $(R, \mathfrak{m}, k)$ is local.

A "semidualizing" R-complex is a homologically finite R-complex X such that the natural morphism $R \to \mathbf{R}\mathrm{Hom}_R(X, X)$ is an isomorphism in the derived category $\mathcal{D}(R)$. In particular, if X is a finitely generated R-module, then it is semidualizing if $\mathrm{Hom}_R(X, X) \cong R$ and $\mathrm{Ext}_R^{\geqslant 1}(X, X) = 0$. These notions were introduced by Foxby [12] and Christensen [9] and, as special cases, recover Grothendieck's dualizing complexes. For some indications of their usefulness, see, e.g., [5, 7, 23, 24].

In [19] we show that R has only finitely many semidualizing complexes up to shift-isomorphism, answering a question of Vasconcelos [26]. The next natural question is: how many semidualizing complexes does a given ring have up to shift-isomorphism? Progress on this question is limited, see, e.g., [8, 22, 25]. As further progress, the main result of this paper is the following, which we prove in 4.3. In the statement, the *embedding codepth* of R is $\mathrm{ecodepth}(R) = \mathrm{edim}(R) - \mathrm{depth}(R)$.

Theorem A. *Let R be a local ring that is Golod or such that* $\mathrm{ecodepth}(R) \leqslant 3$. *Then R has at most two distinct semidualizing complexes up to shift isomorphism, namely, R itself and a dualizing R-complex if one exists.*

The proof uses differential graded algebra techniques, as pioneered by Avramov and his collaborators. It is worth noting that one can prove the Golod case of this result using a result of Jorgensen [17, Theorem 3.1]. However, our approach is different and addresses both cases simultaneously.

Summary. Section 2 consists of foundational material about semidualizing objects in the DG setting. Section 3 contains versions of results from [21] for trivial extensions of DG algebras, including Theorem 3.4 which is key for our proof of Theorem A and may be of independent interest. Section 4 is mainly concerned with the proof of Theorem A.

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Background. We assume that the reader is familiar with many notions from the world of differential graded (DG) algebra. References on the subject include [1, 2, 4, 6, 7, 11, 13, 19, 20]. We most closely follow the conventions from [19]. For the reader's convenience, we specify some terminology and notation.

Complexes of R-modules are indexed homologically. This includes DG algebras and DG modules. Also, our DG algebras are all non-negatively graded. Given a DG R-algebra A, the underlying graded algebra for A is denoted $A^\natural$, and A is *homologically degree-wise noetherian* if $\mathrm{H}_0(A)$ is noetherian and the $\mathrm{H}_0(A)$-module $\mathrm{H}_i(A)$ is finitely generated for all $i \geqslant 0$. When $(R, \mathfrak{m})$ is local, we say that A is *local* if it is homologically degree-wise noetherian and the ring $\mathrm{H}_0(A)$ is a local R-algebra; in this case, the "augmentation ideal" of A is denoted $\mathfrak{m}_A$.

The derived category of DG A-modules is denoted $\mathcal{D}(A)$, and $- \otimes^{\mathbf{L}}_A -$ and $\mathbf{R}\mathrm{Hom}_A(-,-)$ are the derived functors of $- \otimes_A -$ and $\mathrm{Hom}_A(-,-)$. Given an integer i and DG A-modules X and Y, we set $\mathrm{Tor}^A_i(X,Y) := \mathrm{H}_i(X \otimes^{\mathbf{L}}_A Y)$, and we let $\Sigma^i X$ denote the ith *shift* (or *suspension*) of X. Isomorphisms in $\mathcal{D}(A)$ are identified by the symbol $\simeq$, and the DG modules X and Y are "shift-isomorphic", denoted $X \sim Y$, if $X \simeq \Sigma^i Y$ for some $i \in \mathbb{Z}$. We write $\mathrm{id}_A(X) < \infty$ when X has a bounded semi-injective resolution; and $\mathrm{pd}_A(X) < \infty$ means that X has a bounded semi-free resolution. When A is local and X is homologically finite, the *Poincaré series* of X is $P^A_X(t) := \sum_{i\in\mathbb{Z}} \mathrm{len}_k(\mathrm{Tor}^A_i(k,X))t^i$ where $k = A/\mathfrak{m}_A$.

2. Semidualizing DG Modules

Convention. In this section, A is a homologically degree-wise noetherian DG R-algebra.

This section contains some useful DG variations of standard results for semidualizing complexes over rings. We begin with the following definitions from [10, 13].

Definition 2.1. A *semidualizing* DG A-module is a homologically finite DG A-module X such that the natural homothety morphism $R \xrightarrow{\chi^X_A} \mathbf{R}\mathrm{Hom}_A(X,X)$ is an isomorphism in $\mathcal{D}(A)$. When A is a ring concentrated in degree 0, these are "semidualizing A-complexes". The set of shift-isomorphism classes (in $\mathcal{D}(A)$) of semidualizing DG A-modules is denoted $\mathfrak{S}(A)$. A *dualizing* DG A-module is a semidualizing DG A-module D such that for every homologically finite DG A-module M the complex $\mathbf{R}\mathrm{Hom}_A(M,D)$ is homologically finite, and the natural morphisms $M \to \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(M,D),D)$ and $D \otimes^{\mathbf{L}}_A M \to \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(M,R),D)$ are isomorphisms in $\mathcal{D}(A)$. The DG algebra A is *Gorenstein* if A is dualizing.

Next, we summarize some facts about dualizing DG A-modules.

Fact 2.2. Assume that R has a dualizing complex D^R and $A^\natural$ is finitely generated as an R-module. Then $\mathbf{R}\mathrm{Hom}_R(A, D^R)$ is a dualizing DG A-module by [13, Proposition 2.7]. Since $\mathrm{id}_R(D^R) < \infty$, we also have $\mathrm{id}_A(\mathbf{R}\mathrm{Hom}_R(A, D^R)) < \infty$. In particular, every finite dimensional DG algebra over a field has a dualizing DG module of finite injective dimension.

If A is concentrated in degree 0, the next result is well known, and our proof is not surprising; see, e.g., [9, (2.12) Corollary], [14, 2.9.1-2], and [15, Corollary 3.3].

Lemma 2.3. *Assume that A has a dualizing DG module D^A of finite injective dimension, and let X be a semidualizing DG A-module. Then* $\mathbf{R}\mathrm{Hom}_A(X, D^A)$ *is* *a semidualizing DG A-module such that* $X \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A(X, D^A) \simeq D^A$ *in* $\mathcal{D}(A)$*. If* R *and* A *are local and either* $\mathrm{pd}_A(X) < \infty$ *or* $\mathrm{id}_A(A) < \infty$*, then* $X \sim A$*.*

Proof. By [1, Theorem 1], we have

$$X \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A(X, D^A) \simeq \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(X, X), D^A) \simeq \mathbf{R}\mathrm{Hom}_A(A, D^A) \simeq D^A.$$

Next, we have the following commutative diagram of DG A-module morphisms.

$$\begin{array}{ccc} A & \xrightarrow{\chi_A^{\mathbf{R}\mathrm{Hom}_A(X,D^A)}} & \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(X, D^A), \mathbf{R}\mathrm{Hom}_A(X, D^A)) \\ {\scriptstyle \chi_A^{D^A}}\downarrow \simeq & & \downarrow \simeq \\ \mathbf{R}\mathrm{Hom}_A(D^A, D^A) & \xrightarrow[\simeq]{\mathbf{R}\mathrm{Hom}_A(\zeta, D^A)} & \mathbf{R}\mathrm{Hom}_A(X \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A(X, D^A), D^A) \end{array}$$

The morphism ζ is the isomorphism from the beginning of this proof. It follows that $\chi_A^{\mathbf{R}\mathrm{Hom}_A(X,D^A)}$ is an isomorphism, so $\mathbf{R}\mathrm{Hom}_A(X, D^A)$ is a semidualizing.

Assume that R and A are local and either $\mathrm{pd}_A(X) < \infty$ or $\mathrm{id}_A(A) < \infty$. Then by [1, Theorem 1] we have the following isomorphisms in $\mathcal{D}(A)$.

$$X \otimes^{\mathbf{L}}_A \mathbf{R}\mathrm{Hom}_A(X, A) \simeq \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(X, X), A) \simeq A.$$

It follows that $P^A_X(t) P^A_{\mathbf{R}\mathrm{Hom}_A(X,A)}(t) = 1$, so we have $P^A_X(t) = t^d$ for some $d \in \mathbb{Z}$. From this, we conclude that the minimal semi-free resolution $F \xrightarrow{\simeq} X$ has $F^\natural \simeq \Sigma^d A^\natural$. From the Leibniz rule on F, one concludes that $X \simeq F \simeq \Sigma^d A$. $\square$

The next result compares to [10, A.3. Lemma (a)].

Lemma 2.4. *Let* $\underline{t} = t_1, \ldots, t_n$ *be a sequence in the Jacobson radical of* R*, and set* $K = K^R(\underline{t})$*. Let* C *be a DG* R*-algebra such that* $C^\natural$ *is finitely generated over* R*, and set* $B = K \otimes_R C$*. Then for each homologically finite DG* C*-module* X *we have* $X \in \mathfrak{S}(C)$ *if and only if* $B \otimes^{\mathbf{L}}_C X \in \mathfrak{S}(B)$*.*

Proof. Consider the following commutative diagram of chain maps.

$$\begin{array}{ccc} B & \xrightarrow{=} & K \otimes_R C \\ {\scriptstyle \chi_B^{B \otimes^{\mathbf{L}}_C X}}\downarrow & & \downarrow {\scriptstyle K \otimes \chi_C^X} \\ \mathbf{R}\mathrm{Hom}_B(B \otimes^{\mathbf{L}}_C X, B \otimes^{\mathbf{L}}_C X) & & K \otimes_R \mathbf{R}\mathrm{Hom}_C(X, X) \\ \simeq \downarrow & & \\ \mathbf{R}\mathrm{Hom}_C(X, B \otimes^{\mathbf{L}}_C X) & & \downarrow \simeq \\ = \downarrow & & \\ \mathbf{R}\mathrm{Hom}_C(X, (K \otimes_R C) \otimes^{\mathbf{L}}_C X) & \xrightarrow{\simeq} & \mathbf{R}\mathrm{Hom}_C(X, K \otimes^{\mathbf{L}}_R X) \end{array}$$

It follows that $\chi_B^{B \otimes^{\mathbf{L}}_C X}$ is an isomorphism if and only if $K \otimes \chi_C^X$ is an isomorphism. Since C and $\mathbf{R}\mathrm{Hom}_C(X, X)$ are homologically degree-wise finite over R, we conclude (say, from a routine mapping cone argument) that $K \otimes \chi_C^X$ is an isomorphism if and only if χ_C^X is an isomorphism. $\square$

3. Vanishing of Tor over Trivial Extensions of DG Algebras

Convention. In this section, $(R, \mathfrak{m}, k)$ is a local ring.

The point of this section is to prove that Tor-vanishing over certain trivial extensions of DG algebras implies finite projective dimension; see Theorem 3.4. We begin with a useful construction.

Proposition 3.1. *Let (A, A_+) be a local DG R-algebra. Let X be a homologically finite DG A-module. Then there exists a short exact sequence*

$$0 \to X' \xrightarrow{\alpha} L \xrightarrow{\pi} \widetilde{X} \to 0$$

of morphisms of DG A-modules such that L is semi-free with a finite semibasis and we have $\widetilde{X} \simeq X$ and $\operatorname{Im}(\alpha) \subseteq A_+L$.

Proof. Let $F \xrightarrow{\simeq} X$ be a minimal semi-free resolution of X, and let E be a semibasis for F. Let $F^{(p)}$ be the semi-free DG A-submodule of F spanned by $E_{\leqslant p} := \bigcup_{m \leqslant p} E_m$. To be clear, $F^{(p)\natural}$ is the graded submodule of $F^\natural$ spanned by $E_{\leqslant p}$. Note that $\partial^F(F^{(p)\natural}) \subseteq F^{(p)\natural}$. To see this, by the Leibniz rule we have $\partial^F_{i+j}(te) = \partial^A_i(t)e + (-1)^i t\partial^F_j(e)$ for each $t \in A$ of degree i and each $e \in F$ of degree j. If $j \leqslant p$, then the term $\partial^A_i(t)e$ is in $\operatorname{Span}_A(E_{\leqslant p})$ by assumption, and the term $(-1)^i t\partial^F_j(e)$ is in $\operatorname{Span}_A(E_{\leqslant p})$ by a degree argument.

Let $s = \sup(X)$, and set $\widetilde{X} = \tau_{\leqslant s}(F)$, the "soft truncation" of F. Note that the natural morphism $F \to \widetilde{X}$ is a surjective quasiisomorphism of DG A-modules, so we have $\widetilde{X} \simeq F \simeq X$. Next, set $L = F^{(s)}$, which is semi-free with a finite semibasis $E_{\leqslant s}$. Furthermore, the composition π of natural morphisms $L = F^{(s)} \to F \to \widetilde{X}$ is surjective because the morphism $F \to \widetilde{X}$ is surjective, the morphism $L \to F$ is surjective in degrees $\leqslant s$, and we have $\widetilde{X}_i = 0$ for all $i > s$. Thus, it remains to show that $X' := \operatorname{Ker}(\pi) \subseteq A_+L$.

By construction, for $i \leqslant s$, the map π_s is an isomorphism, so $X'_i = 0 \subseteq A_+L$, as desired. In degree $s+1$, we have $X'_{s+1} = \operatorname{Im}(\partial^F_{s+1})$, which is contained in $(A_+F)_s$ by construction. Also, since A_+ consists of elements of positive degree, we have

$$(A_+F)_s \subseteq A_+ \operatorname{Span}_A(E_{<s}) \subseteq A_+F^{(s)}.$$

Thus, we have $X'_{s+1} \subseteq A_+F^{(s)}$. Lastly, for $i > s$, we have $X'_i = F^{(s)}_i = (A_+F^{(s)})_i$ by a degree argument, since $F^{(s)}$ is generated over A in degrees $\leqslant s$. □

Corollary 3.2. *Let (A, A_+) be a bounded local DG R-algebra. Let X and Y be homologically finite DG A-modules of infinite projective dimension over A. Assume that $\operatorname{Tor}^A_{i \gg 0}(X, Y) = 0$. Then there are homologically finite DG A-modules X', Y' of infinite projective dimension over A such that $\operatorname{Ann}_A(A_+)X' = 0 = \operatorname{Ann}_A(A_+)Y'$ and $\operatorname{Tor}^A_{i \gg 0}(X', Y') = 0$.*

Proof. By Proposition 3.1, there are short exact sequences

$$0 \to X' \xrightarrow{\alpha} L \to \widetilde{X} \to 0$$

$$0 \to Y' \xrightarrow{\beta} M \to \widetilde{Y} \to 0$$

of morphisms of DG A-modules such that L and M are semi-free with a finite semibases, and $\widetilde{X} \simeq X$, $\widetilde{Y} \simeq Y$, $\operatorname{Im}(\alpha) \subseteq A_+L$, and $\operatorname{Im}(\beta) \subseteq A_+M$. The condition

$\operatorname{Im}(\alpha) \subseteq A_+L$ implies that

$$\operatorname{Ann}_A(A_+)X' \cong \operatorname{Ann}_A(A_+)\operatorname{Im}(\alpha) \subseteq \operatorname{Ann}_A(A_+)A_+L = 0$$

and similarly $\operatorname{Ann}_A(A_+)Y' = 0$. By assumption, $\operatorname{pd}_A(L) < \infty$, so the condition $\operatorname{pd}_A(\widetilde{X}) = \operatorname{pd}_A(X) = \infty$ implies that $\operatorname{pd}_A(X') = \infty$, because of the exact sequence $0 \to X' \to L \to \widetilde{X} \to 0$. And similarly $\operatorname{pd}_A(Y') = \infty$. Finally, since A is bounded and $\operatorname{pd}_A(L)$ is finite, the fact that Y is homologically bounded implies that $\operatorname{Tor}^A_{i\gg 0}(L,Y) = 0$. By assumption, we have $\operatorname{Tor}^A_{i\gg 0}(\widetilde{X},Y) \cong \operatorname{Tor}^A_{i\gg 0}(X,Y) = 0$ so the above exact sequence implies that $\operatorname{Tor}^A_{i\gg 0}(X',Y) = 0$. Similarly, we deduce $\operatorname{Tor}^A_{i\gg 0}(X',Y') = 0$, as desired. □

Compare the next two results to [21, Lemma 3.2 and Theorem 3.1].

Lemma 3.3. *Let $(B,\mathfrak{m}_B,k)$ be a local DG R-algebra. Set $A = B \ltimes \Sigma^n k$ for some $n \geqslant 0$, and let $\mathfrak{m}_A$ be the augmentation ideal of A. Let x be a generator for the DG ideal $0 \oplus \Sigma^n k \subseteq A$. Let X, Y be DG B-modules, i.e., DG A-modules such that $xX = 0 = xY$. Then for all $i \in \mathbb{Z}$ we have R-module isomorphisms*

$$\operatorname{Tor}^A_i(X,Y) \cong \operatorname{Tor}^B_i(X,Y) \bigoplus \left(\bigoplus_{p+q=i-n-1} \operatorname{Tor}^A_p(X,k) \otimes_k \operatorname{Tor}^B_q(k,Y) \right).$$

Proof. Let $s\colon L \xrightarrow{\simeq} B$ be a semi-free resolution over A. Let $p\colon A \to B$ be the natural surjection, and let $\widetilde{p}\colon A \to L$ be a lift of p. Hence the following diagram of morphisms of DG A-modules

$$\begin{array}{ccc} A & \xrightarrow{\widetilde{p}} & L \\ & \searrow^{p} & \downarrow{\scriptstyle s}\,\simeq \\ & & B \end{array}$$

commutes up to homotopy. Apply $X \otimes_A -$ to obtain the next diagram of morphisms of DG B-modules

$$\begin{array}{ccc} X \otimes_A A & \xrightarrow{X\otimes\widetilde{p}} & X \otimes_A L \\ & \searrow^{X\otimes p}_{\cong} & \downarrow{\scriptstyle X\otimes s} \\ & & X \otimes_A B \end{array}$$

that commutes up to homotopy. Note that $X \otimes p$ is an isomorphism since $xX = 0$. Also, the chain map $X \otimes_A \widetilde{p}$ represents the morphism $X \otimes^{\mathbf{L}}_A p\colon X \otimes^{\mathbf{L}}_A A \to X \otimes^{\mathbf{L}}_A B$ in $\mathcal{D}(B)$. It follows that $X \otimes^{\mathbf{L}}_A p$ has a left-inverse in $\mathcal{D}(B)$, so we have the first isomorphism (in $\mathcal{D}(B)$) in the next sequence:

$$\begin{aligned} X \otimes^{\mathbf{L}}_A B &\simeq \left(X \otimes^{\mathbf{L}}_A A\right) \bigoplus \left(X \otimes^{\mathbf{L}}_A (\Sigma xA)\right) \\ &\simeq \left(X \otimes^{\mathbf{L}}_A A\right) \bigoplus \left(X \otimes^{\mathbf{L}}_A (\Sigma^{n+1} k)\right) \\ &\simeq \left(X \otimes^{\mathbf{L}}_A A\right) \bigoplus \left(\Sigma^{n+1} X \otimes^{\mathbf{L}}_A k\right). \end{aligned}$$

The second isomorphism is from the assumption $\mathfrak{m}_A x = 0$. (Note that this isomorphism is in $\mathcal{D}(A)$, hence also in $\mathcal{D}(B)$ via the split injection $B \to A$.) Now, we apply $-\otimes^{\mathbf{L}}_B Y$ to these isomorphisms to conclude that

$$\begin{aligned} X\otimes^{\mathbf{L}}_A Y &\simeq (X\otimes^{\mathbf{L}}_A B)\otimes^{\mathbf{L}}_B Y \\ &\simeq \left((X\otimes^{\mathbf{L}}_A A)\otimes^{\mathbf{L}}_B Y\right)\bigoplus\left(\Sigma^{n+1}(X\otimes^{\mathbf{L}}_A k)\otimes^{\mathbf{L}}_B Y\right) \\ &\simeq \left(X\otimes^{\mathbf{L}}_B Y\right)\bigoplus\left(\Sigma^{n+1}(X\otimes^{\mathbf{L}}_A k)\otimes^{\mathbf{L}}_k (k\otimes^{\mathbf{L}}_B Y)\right). \end{aligned}$$

Apply $\mathrm{H}_i(-)$ to obtain the first isomorphism in the next sequence:

$$\begin{aligned} \operatorname{Tor}^A_i(X,Y) &\cong \operatorname{Tor}^B_i(X,Y)\bigoplus \mathrm{H}_i\left(\Sigma^{n+1}(X\otimes^{\mathbf{L}}_A k)\otimes^{\mathbf{L}}_k (k\otimes^{\mathbf{L}}_B Y)\right) \\ &\cong \operatorname{Tor}^B_i(X,Y)\bigoplus \mathrm{H}_{i-n-1}\left((X\otimes^{\mathbf{L}}_A k)\otimes^{\mathbf{L}}_k (k\otimes^{\mathbf{L}}_B Y)\right) \\ &\cong \operatorname{Tor}^B_i(X,Y)\bigoplus\left(\bigoplus_{p+q=i-n-1}\mathrm{H}_p(X\otimes^{\mathbf{L}}_A k)\otimes_k \mathrm{H}_q(k\otimes^{\mathbf{L}}_B Y)\right) \\ &\cong \operatorname{Tor}^B_i(X,Y)\bigoplus\left(\bigoplus_{p+q=i-n-1}\operatorname{Tor}^A_p(X,k)\otimes_k \operatorname{Tor}^B_q(k,Y)\right). \end{aligned}$$

The third isomorphism is from the Künneth formula. □

Theorem 3.4. *Let (B,B_+,k) be a bounded local DG R-algebra. Set $A=B\ltimes\Sigma^n k$ for some $n\geqslant 1$. Let x be a generator for the DG ideal $0\oplus\Sigma^n k\subseteq A$. Let X and Y be non-zero homologically finite DG A-modules such that $\operatorname{Tor}^A_{i\gg 0}(X,Y)=0$. Then $\operatorname{pd}_A(X)<\infty$ or $\operatorname{pd}_A(Y)<\infty$.*

Proof. Assume by way of contradiction that $\operatorname{pd}_A(X)=\infty=\operatorname{pd}_A(Y)$. Thus, Corollary 3.2 provides homologically finite DG A-modules X',Y' of infinite projective dimension such that $\operatorname{Ann}_A(A_+)X'=0=\operatorname{Ann}_A(A_+)Y'$ and such that $\operatorname{Tor}^A_{i\gg 0}(X',Y')=0$. Thus, we may replace X and Y by X' and Y' to assume that $\operatorname{Ann}_A(A_+)X=0=\operatorname{Ann}_A(A_+)Y$. In particular, we have $xX=0=xY$.

Since $\operatorname{Tor}^A_{i\gg 0}(X,Y)=0$, Lemma 3.3 implies that

$$\bigoplus_{p+q=i-n-1}\operatorname{Tor}^A_p(X,k)\otimes_k\operatorname{Tor}^B_q(k,Y)=0 \tag{3.4.1}$$

for all $i\gg 0$. The fact that $\operatorname{pd}_A(Y)$ is infinite implies that $Y\not\simeq 0$. Since Y is homologically finite, Nakayama's Lemma implies that there is an integer q_0 such that $\operatorname{Tor}^B_{q_0}(k,Y)\neq 0$. Thus, equation (3.4.1) for $i\gg 0$ implies that $\operatorname{Tor}^A_{p\gg 0}(X,k)=0$, contradicting the assumption $\operatorname{pd}_A(X)=\infty$. □

4. The Number Of Semidualizing Complexes For Embedding Codepth 3

Convention. In this section, $(R,\mathfrak{m},k)$ is a local ring.

The purpose of this section is to prove Theorem A from the introduction, which we do in 4.3. Our main tool for this is the following result.

Theorem 4.1. *Assume that R admits a dualizing complex, and let (B,B_+,k) be a bounded homologically finite local DG R-algebra. Set $A=B\ltimes W$ for some non-zero finitely generated positively graded k-vector space W. Then $|\mathfrak{S}(A)|\leqslant 2$.*

Proof. Fact 2.2 implies that A has a dualizing DG module D^A of finite injective dimension. Set $(-)^\dagger = \mathbf{R}\mathrm{Hom}_A(-, D^A)$. It suffices to show that for every semidualizing DG A-module X we have $X \sim A$ or $X \sim D^A$.

Case 1: $W = \Sigma^n k$ for some $n \geqslant 1$. The isomorphism $X \otimes^{\mathbf{L}}_A X^\dagger \simeq D^A$ from Lemma 2.3 implies that $\mathrm{Tor}^A_{i \gg 0}(X, X^\dagger) = 0$. By Theorem 3.4 either $\mathrm{pd}_A(X) < \infty$ or $\mathrm{pd}_A(X^\dagger) < \infty$. And by Lemma 2.3 we have $X \sim A$ or $X^\dagger \sim A$. If $X^\dagger \sim A$, then by definition of D^A we have $X \simeq X^{\dagger\dagger} \sim A^\dagger \simeq D^A$, as desired.

Case 2: General case. Since $W \neq 0$, write $W = W' \oplus \Sigma^n k$ for some $n \geqslant 1$, and set $B' := B \ltimes W'$. It follows that $A \cong B' \ltimes \Sigma^n k$, so the assertion follows from the previous case. □

Remark 4.2. Let $\underline{t} = t_1, \cdots, t_n$ be a minimal generating set for $\mathfrak{m}$, and consider the Koszul complex $K = K^R(\underline{t})$. From [3, (1.2)] (or [4, (2.8)]) there is a finite-dimensional DG k-algebra A that is linked to K by a sequence of quasiisomorphisms of DG algebras. As in [19, 5.4 (Proof of Theorem A)], one has an injection $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}(\widehat{R})$ and bijections $\mathfrak{S}(\widehat{R}) \xrightarrow{\cong} \mathfrak{S}(K) \xrightarrow{\cong} \mathfrak{S}(A)$.

By definition (or by a result of Golod [16]), R is Golod if and only if A is of the form $k \ltimes W$ for some finitely generated positively graded k-vector space W.

Assume for the rest of this remark that $c = \mathrm{ecodepth}(R) \leqslant 3$. Up to isomorphism, the algebra A has $\partial^A = 0$ and is in one of the classes described in the next table, copied from [3, 1.3].

Class	A	B	C	D
$\mathbf{C}(c)$	B	$\bigwedge_k \Sigma k^c$		
$\mathbf{S}$	$B \ltimes W$	k		
$\mathbf{T}$	$B \ltimes W$	$C \ltimes \Sigma(C/C_{\geqslant 2})$	$\bigwedge_k \Sigma k^2$	
$\mathbf{B}$	$B \ltimes W$	$C \ltimes \Sigma C_+$	$\bigwedge_k \Sigma k^2$	
$\mathbf{G}(r)$	$B \ltimes W$	$C \ltimes \mathrm{Hom}_k(C, \Sigma^3 k)$	$k \ltimes \Sigma k^r$	
$\mathbf{H}(p,q)$	$B \ltimes W$	$C \otimes_k D$	$k \ltimes (\Sigma k^p \oplus \Sigma^2 k^q)$	$k \ltimes \Sigma k$

Here, W is a finitely generated positively graded k-vector space such that $B_+ W = 0$. Note that if R is not regular, then $\sum_i (-1)^i \mathrm{len}_k(A_i) = \sum_i (-1)^i \mathrm{len}_k(\mathrm{H}_i(K)) = 0$.

4.3 (Proof of Theorem A)**.** Continue with the notation from Remark 4.2. Recall that $\widehat{R}$ has a dualizing complex $D^{\widehat{R}}$, and that R is Gorenstein if and only if $D^{\widehat{R}} \sim \widehat{R}$. Hence, it suffices to show that $|\mathfrak{S}(A)| \leqslant 2$.

Assume for this paragraph that R is Golod. Then we have $A \cong k \ltimes W$ for some finitely generated positively graded k-vector space W. If $W = 0$, then $A \cong k$ which is a commutative local Gorenstein ring, hence $|\mathfrak{S}(A)| = 1$ in this case. If $W \neq 0$, then $|\mathfrak{S}(A)| \leqslant 2$ by Theorem 4.1.

Assume for the rest of the proof that $c = \mathrm{ecodepth}(R) \leqslant 3$. We analyze the classes from Remark 4.2. Note that if R is Gorenstein, then $|\mathfrak{S}(R)| = 1$. Also, if $W \neq 0$, then the conclusion $|\mathfrak{S}(A)| \leqslant 2$ follows from Theorem 4.1. Thus, we assume for the rest of the proof that $W = 0$.

If R is in the class $\mathbf{C}(c)$, then R is a complete intersection (hence Gorenstein), which has already been treated. If R is in the class $\mathbf{S}$, then R is Golod, which has also already been treated.

(Class $\mathbf{T}$) In this case the algebra C has the form $0 \to k \to k^2 \to k \to 0$. In particular, $C_{\geqslant 2} = \Sigma^2 k$. It follows that

$$\sum_i (-1)^i \operatorname{len}_k(B_i) = \sum_i (-1)^i \operatorname{len}_k(C_i) + \sum_{i<2} (-1)^{i-1} \operatorname{len}_k(C_i) = 1 \neq 0.$$

Since we know that

$$\begin{aligned} 0 &= \sum_i (-1)^i \operatorname{len}_k(A_i) \\ &= \sum_i (-1)^i \operatorname{len}_k(B_i) + \sum_i (-1)^i \operatorname{len}_k(W_i) \\ &= 1 + \sum_i (-1)^i \operatorname{len}_k(W_i) \end{aligned}$$

we have $W \neq 0$, which is a case we have already treated.

(Class $\mathbf{B}$) As in the previous case, one has $W \neq 0$, which has already been treated.

(Class $\mathbf{G}$(r)) The DG C-module $\operatorname{Hom}_k(C, \Sigma^3 k)$ is dualizing, so the algebra

$$A \cong B = C \ltimes \operatorname{Hom}_k(C, \Sigma^3 k)$$

is Gorenstein by [18, Theorem 2.2]. It follows from [13, Theorem III] that $A \simeq \operatorname{Hom}_k(A, \Sigma^3 k)$. Since $\operatorname{Hom}_k(A, \Sigma^3 k)$ is bounded and semi-injective over A, we conclude that $\operatorname{id}_A(A) < \infty$. Thus, we have $|\mathfrak{S}(A)| = 1$ by Lemma 2.3.

(Class $\mathbf{H}$(p,q)) We first note that $|\mathfrak{S}(C)| \leqslant 2$. Indeed, if $p = q = 0$, then we have $C = k$ so $|\mathfrak{S}(C)| = 1$, as above; if $p \neq 0$ or $\mathfrak{q} \neq 0$, then this follows from Theorem 4.1. Hence, it remains to show that $|\mathfrak{S}(B)| = |\mathfrak{S}(C)|$. For this, consider the map $\mathfrak{S}(C) \to \mathfrak{S}(B)$ defined by $X \mapsto B \otimes^{\mathbf{L}}_C X$. This is well-defined by Lemma 2.4, and it is bijective by [20, Theorems 3.4 and 3.11]. (This uses the fact that D is isomorphic to the trivial Koszul complex $K^k(0)$.) □

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Department of Mathematics, University of Nebraska–Lincoln, Lincoln, NE 68588-0130, USA
E-mail address: `snasseh2@unl.edu`
URL: `http://www.math.unl.edu/~snasseh2/`

Department of Mathematics, North Dakota State University Dept # 2750, PO Box 6050, Fargo, ND 58108-6050 USA
E-mail address: `sean.sather-wagstaff@ndsu.edu`
URL: `http://www.ndsu.edu/pubweb/~ssatherw/`