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Tight super-edge-graceful labelings of trees and their applications

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Abstract

The concept of graceful labeling of graphs has been extensively studied. In 1994, Mitchem and Simoson introduced a stronger concept called super-edge-graceful labeling for some classes of graphs. Among many other interesting pioneering results, Mitchem and Simoson provided a simple but powerful recursive way of constructing super-edge-graceful trees of odd order. In this note, we present a stronger concept of "tight" super-edge-graceful labeling. Such a super-edge graceful labeling has an additional constraint on the edge and vertices with the largest and smallest labels. This concept enables us to recursively construct tight super-edge-graceful trees of any order. As applications, we provide insights on the characterization of super-edge-graceful trees of diameter 4, a question posed by Chung, Lee, Gao and Schaffer. We also observe infinite families of super-edge-graceful trees that can be generated from tight labelings. Given the direct applications of "tight" super-edge-graceful labeling to the study of super-edge-graceful labelings, we note that it is worthwhile to further examine recursively generated tight super-edge-graceful trees.

Keywords: Graceful labeling; Super-edge-graceful; Trees

1. Introduction

A finite, simple and undirected graph G with $p = |V(G)|$ and $q = |E(G)|$ is called *edge-graceful* if there are onto functions

$$l : E(G) \to \{1, \ldots, q\}$$

and

$$l^* : V(G) \to \{1, \ldots, p\}$$

such that

$$l^*(v) = \sum_{uv \in E(G)} l(uv) \bmod p.$$

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When it exists, such a labeling is called an *edge-graceful labeling*. Introduced by Lo [1] in 1985, this concept has been extensively studied. In particular, many classes of trees of odd orders were shown to be edge-graceful [2–6] as partial results towards the following conjecture:

Conjecture 1.1 ([7]). *All odd-order trees are edge-graceful.*

In [3], all trees of odd order of diameter at most four have been shown to be edge-graceful. The concept of *super-edge-graceful* graphs was introduced in [4].

Definition 1. A graph G of order p and size q is super-edge-graceful (SEG) if there exists a bijection l from $E(G)$ to

$$\left\{0, \pm 1, \ldots, \pm \left\lfloor \frac{q}{2} \right\rfloor\right\} \quad \text{if } q \text{ is odd}$$

and

$$\left\{\pm 1, \ldots, \pm \left\lfloor \frac{q}{2} \right\rfloor\right\} \quad \text{if } q \text{ is even}$$

such that the induced vertex labeling l^*

$$l^*(v) = \sum_{uv \in E(G)} l(uv)$$

is a bijection from $V(G)$ to

$$\left\{0, \pm 1, \ldots, \pm \left\lfloor \frac{p}{2} \right\rfloor\right\} \quad \text{if } p \text{ is odd}$$

and

$$\left\{\pm 1, \ldots, \pm \left\lfloor \frac{p}{2} \right\rfloor\right\} \quad \text{if } p \text{ is even.}$$

When it exists, such a labeling is called a *super-edge-graceful labeling* (SEGL). For trees, it is shown in [4] that SEG implies edge-graceful. Also in [4], the following simple but extremely useful "growing tree algorithm" was provided.

Proposition 1.2 ([4]). *By appending two edges to the same vertex of a SEG tree of odd order, one obtains a new tree that is also SEG.*

For a list of the work done on various graph labelings, we recommend the dynamic survey [8]. As a related question to edge-graceful trees, the following was posed in [9]:

Question 1.1 ([9]). *Characterize trees of diameter* 4 *that are SEG.*

All trees of odd order with three even vertices are shown to be SEG in [10]. Most recently, [11] provided explicit SEGL for *caterpillars* (trees whose removal of leaves results in a path) and some *lobsters* (trees whose removal of leaves results in a caterpillar) of diameter 4.

In this note, we present a generalization of SEGL that we call a *tight super-edge-graceful labeling* (tight SEGL). The precise definition of a tight SEGL is provided in Section 2, where we also show a generalized version of Proposition 1.2 that applies to trees of even order. As applications, we show, in Section 3, how Question 1.1 can be reduced to simpler trees through the utilization of tight SEGL. We also make some modest progress towards some open questions from [11] through this approach. In Section 4, we briefly mention how our recursive construction can be used to generate an infinite family of trees (of even order) that have tight SEGL.

2. Tight labeling and the "growing tree algorithm" for even ordered trees

Towards our goal of providing a generalization of Proposition 1.2 for trees of even order, we first define the *tight SEGL* for trees of even order.

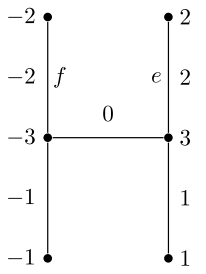

Fig. 1. Example of a tight SEGL.

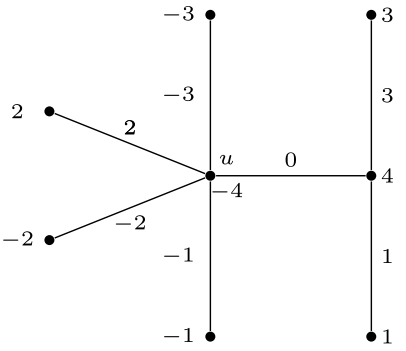

Fig. 2. Example of a tight SEGL generated from Proposition 2.1.

Definition 2. A SEGL of a tree T of even order is said to be "tight" if there exist $e = ab$, $f = cd \in E(T)$ such that

$$l(e) = \left\lfloor \frac{q}{2} \right\rfloor \quad \text{and} \quad l(f) = -\left\lfloor \frac{q}{2} \right\rfloor$$

with the vertex labelings

$$\{l^*(a), l^*(b)\} = \left\{ \left\lfloor \frac{p}{2} \right\rfloor, \left\lfloor \frac{p}{2} \right\rfloor - 1 \right\} \quad \text{and} \quad \{l^*(c), l^*(d)\} = \left\{ -\left\lfloor \frac{p}{2} \right\rfloor, -\left\lfloor \frac{p}{2} \right\rfloor + 1 \right\}.$$

For example, Fig. 1 shows the tight SEGL of a tree of order 6 with e and f labeled as ± 2.

Note that the tight labeling of Definition 2 can also be defined for trees of odd order. Such a variation would be meaningless though since the purpose of tight SEGLs is to provide an analogue of Proposition 1.2 as follows.

Proposition 2.1 (*Growing Tree Algorithm for Even Ordered Trees*)**.** *By appending two edges to the same vertex of a tight SEG tree of even order, one obtains a new tree that also has a tight SEGL.*

Proof. Let T be a tree of even order $|V(T)| = 2n$ with a tight SEGL. Then the tight SEGL of T yields two edges $e = ab$ and $f = cd$ with

$$l(e) = n - 1; \qquad l(f) = -(n-1); \qquad l^*(a) = n;$$
$$l^*(b) = n - 1; \qquad l^*(c) = -n; \qquad l^*(d) = -(n-1).$$

Let T' be a tree obtained by appending two edges uv and uw to the same vertex u. Then $|V(T')| = 2n + 2$. We obtain a tight SEGL by keeping the labelings of all vertices/edges in T except for the following:

$$l(e) = n; l(f) = -n; \qquad l^*(a) = n + 1; \qquad l^*(b) = n; \qquad l^*(c) = -(n+1); \qquad l^*(d) = -n.$$

The new vertices and edges, we label as follows:

$$l^*(v) = l(uv) = n - 1; \qquad l^*(w) = l(uw) = -(n-1). \quad \blacksquare$$

For example, Fig. 2 shows a tight SEG tree generated from Fig. 2 by applying Proposition 2.1.

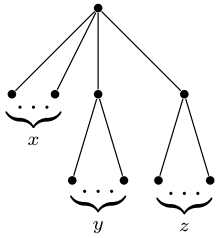

Fig. 3. A caterpillar of diameter 4 corresponding to (x, y, z).

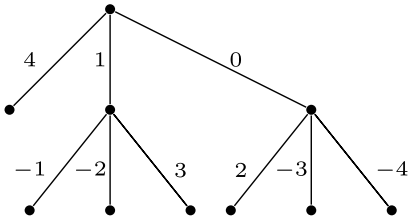

Fig. 4. A tight SEGL of the caterpillar $(1, 3, 3)$.

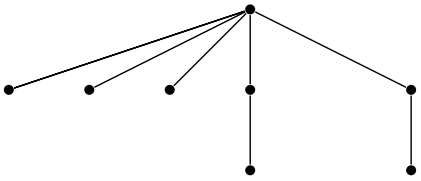

Fig. 5. A spider with 2 legs of length 2 and 3 legs of length 1.

3. On trees of diameter 4

We restrict our attention to Question 1.1 and trees of diameter four in this section.

First of all, it is easy to see that a tree of diameter four is either a caterpillar or a lobster [11]. As nicely demonstrated in [11], caterpillars of diameter four can be represented as a three-tuple (x, y, z) (Fig. 3).

If such a caterpillar is of even order, then $x + y + z$ is odd. Take, for example, the case of $x \geq 1$, $y \geq 3$ and $z \geq 3$ each of which being odd. Such a tree can be easily reduced to $(x, y, z) = (1, 3, 3)$ by repeatedly removing two adjacent pendant edges. Fig. 4 displays a tight SEGL of such a tree. Then by applying Proposition 2.1, one can generate a tight SEGL (and hence SEGL) for any tree of this kind.

Remark 1. This approach of "growing tree algorithm" through Proposition 2.1 (for even ordered trees) and Proposition 1.2 (for odd ordered trees) can be applied to all similar cases. We omit the details since explicit labelings were presented in [11], in which the set of all caterpillars of diameter four that possess a SEGL are characterized.

Given any lobster of diameter 4, by following the same approach, the tree can be reduced to a *spider* (a tree with exactly one vertex of degree > 2) with legs of length 1 or 2 (Fig. 5).

Unfortunately, such graphs do not necessarily yield a SEGL or tight SEGL. However, by adding a few edges, one can obtain some "semi-elementary" trees that do have a tight SEGL. See, for instance, Fig. 6.

In [11], the explicit labelings of many lobsters of diameter 4 are presented. We would like to mention that they can also be generated from elementary cases, such as Fig. 6, through the "growing tree algorithms". For the few remaining cases, there are descriptions in Conjectures 3.12–3.14 in [11]. We skip the details of their descriptions but point out that the questions from Conjectures 3.13 and 3.14 can be reduced to the following simpler structure but stronger statement.

Question 3.1. *Consider a tree T of height 2 rooted at v with $k + 2$ children $v_0, v_1, \ldots, v_k, u$ such that v_0 is a leaf, v_i is of degree 2 for $1 \leq i \leq k$ and u is of degree 4. Does there always exist a tight SEGL of T for $k \geq 2$? For example, Fig. 6 shows such a tree with $k = 2$.*

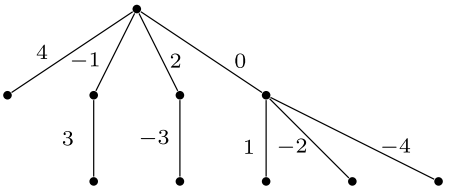

Fig. 6. The tight SEGL of a small lobster.

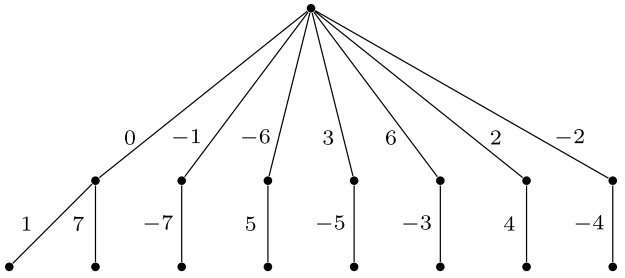

Fig. 7. A tight SEGL for $k = 3$ in Question 3.2.

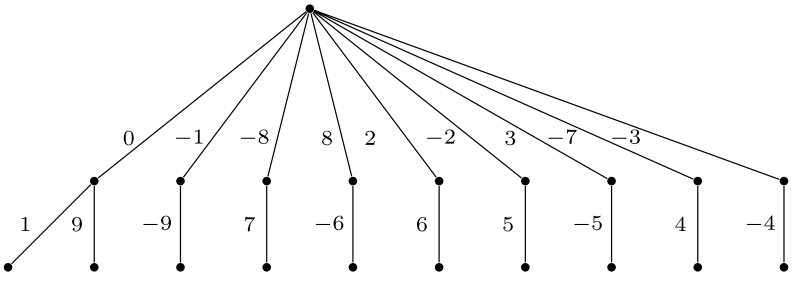

Fig. 8. A tight SEGL for $k = 4$ in Question 3.2.

Remark 2. A positive answer to Question 3.1 will imply a tight SEGL for any tree generated from them through the "growing tree algorithm", among them the ones conjectured in [11].

We have yet to find a general labeling for such trees. However, computer simulations yield a positive answer to Question 3.1 for all small values of k. With such evidence at hand, we conjecture that this question can be answered positively.

In an analogous way, Conjecture 3.12 of [11] can be reduced to the following stronger statement on much simpler structures.

Question 3.2. *Consider a tree T of height 2 rooted at v with $2k + 1$ children $v_0, v_1, \ldots, v_{2k}$ such that v_0 is of degree 3 and v_i is of degree 2 for $1 \leq i \leq 2k$. Does there always exist a tight SEGL of T for $k \geq 3$? For example,* Figs. 7 and 8 *show such trees with $k = 3, 4$.*

Remark 3. When $k = 1$ in Question 3.2, it can be seen that such a tree does not have a SEGL. When $k = 2$, a SEGL can be easily found but no tight SEGL exists.

4. Trees with tight super-edge-graceful labelings

The "growing tree algorithm" can easily generate infinite families of tight SEG trees of even order. For instance, every tree shown in Figure 3 of [10] of order 8 that has a SEGL also has a tight SEGL.

By applying Proposition 2.1 to these trees, we easily get various trees with tight SEGLs. It seems to be an interesting question to try to provide some characterization of tight SEG trees generated through this venue.

Acknowledgment

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References

[1] S.P. Lo, On edge-graceful labellings of graphs, Congr. Numer. 50 (1985) 231–241.
[2] S. Cabannis, J. Michem, R. Low, On edge-graceful regular graphs and trees, Ars Combin. 34 (1992) 129–142.
[3] S.M. Lee, K. Nowak, L. Wang, W. Wei, On the edge-graceful trees conjecture, J. Combin. Math. Combin. Comput. (2015) in press.
[4] J. Mitchem, A. Simoson, On edge-graceful and super-edge-graceful graphs, Ars Combin. 37 (1994) 97–111.
[5] A. Simoson, Edge-graceful cootie, Congr. Numer. 101 (1994) 117–128.
[6] D. Small, Regular (even) spider graphs are edge-graceful, Congr. Numer. 74 (1990) 247–254.
[7] S.M. Lee, A conjecture on edge-graceful trees, Sci. Ser. A. 3 (1989) 45–57.
[8] J.A. Gallian, A dynamic survey of graph labeling, Electron. J. Combin. (2014).
[9] P.T. Chuang, S.M. Lee, W.Y. Gao, K. Schaffer, On the super edge graceful trees of even orders, Congr. Numer. 181 (2006) 5–17.
[10] S.M. Lee, Y.S. Ho, All trees of odd order with three even vertices are super edge-graceful, J. Combin. Math. Combin. Comput. 62 (2007) 53–64.
[11] E. Krop, F. Mutiso, C. Raridan, On super edge-graceful trees of diameter four, preprint.