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Yuan Liang
Zhejiang Wanli University, ly0432@163.com

Xianbin Wu
Zhejiang Wanli University

Qihu Zhang
Zhengzhou University of Light Industry

Chunshan Zhao
Georgia Southern University, czhao@georgiasouthern.edu

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MULTIPLE SOLUTIONS OF A $p(x)$-LAPLACIAN EQUATION INVOLVING CRITICAL NONLINEARITIES

Yuan Liang, Xianbin Wu*, Qihu Zhang* and Chunshan Zhao

Abstract. In this paper, we consider the existence of multiple solutions for the following $p(x)$-Laplacian equations with critical Sobolev growth conditions

$$\begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u)+|u|^{p(x)-2}u=f(x,u) \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega. \end{cases}$$

We show the existence of infinitely many pairs of solutions by applying the Fountain Theorem and the Dual Fountain Theorem respectively. We also present a variant of the concentration-compactness principle, which is of independent interest.

1. INTRODUCTION

In recent years, there are a lot of interest in the study of various mathematical problems with variable exponent (see [2-6, 13, 14, 16-25, 30-34, 36-39, 42, 44-51, 54-58] and references therein). We refer readers to [17, 51] for an overview of this research area and [2, 13, 49, 58] for the background of these problems. Recently, people are also interested in the applications of variable exponent analysis to image restoration [29, 30, 34, 38]. The most typical differential equation with variable exponent is the $p(x)$-Laplacian equation, which is a generalization of the usual p-Laplacian equation with the constant exponent p being replaced by a variable exponent $p(x)$. For Sobolev spaces with variable exponent which have been used to study the $p(x)$-Laplacian equations, we refer readers to [16, 19, 37]. On the existence of solutions of elliptic equations with variable exponent and subcritical growth conditions, we refer readers to [5, 6, 21,

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*Corresponding author.

23, 31, 54]. However, to the best of our knowledge, results on elliptic equations with variable exponent and critical growth condition are rare (see [24, 25]).

In this paper, we consider the existence of multiple solutions of the following equations with critical Sobolev growth conditions

$$\text{(P)} \qquad \begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = f(x,u) \ \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\triangle_{p(x)}u := div(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian; $\Omega \subset \mathbb{R}^N$ is an open bounded dommain, $p(x) \in C(\overline{\Omega})$ is Lipschitz continuous and $1 < \inf_{x\in\Omega} p(x) \leq \sup_{x\in\Omega} p(x) < N$; f satisfies the following condition.

$$|f(x,t)| \leq C(1 + |t|^{p^*(x)-1}), \forall (x,t) \in \overline{\Omega} \times \mathbb{R},$$

where $p^*(x) = \begin{cases} Np(x)/(N-p(x)), p(x) < N, \\ \infty, p(x) \geq N. \end{cases}$

Because of its non-homogeneity, the $p(x)$-Laplacian possesses more complicated nonlinearity than the p-Laplacian. Many results for p-Laplacian problems do not hold for $p(x)$-Laplacian problems anymore. For examples,

(1^0) If $\Omega \subset \mathbb{R}^N$ is an open bounded domain, the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(\cdot)}(\Omega)\backslash\{0\}} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx}$$

is zero in general. Only under some special conditions, we have $\lambda_{p(x)} > 0$. For example, $\lambda_{p(x)} > 0$ if and only if $p(x)$ is monotone in one dimensional case (i.e. $N = 1$) (see [22]). It is well known that the fact that $\lambda_p > 0$ is very important in the study of p-Laplacian problems.

(2^0) The norm in $L^{p(\cdot)}(\Omega)$ is of Luxemburg type (we will explain later in the second section). It is easy to see that $\int_\Omega |u|^{p(x)}\, dx = |u|_{p(\cdot)}^{p(\xi)}$ for some $\xi \in \overline{\Omega}$. Hence the integral and the norm can not keep the constant exponent relationship. It implies that we will have more difficulties in the study of $p(x)$-Laplacian problems. For example, it is very difficult to get the best Sobolev imbedding constant when we deal with the critical Sobolev exponent problems. Even if the best Sobolev imbedding constant could be obtained, it is also hard to be applied to study the critical exponent problems.

In [8], Brézis and Nirenberg initially studied the equations involving critical Sobolev exponents. In [40, 41], Lions discovered the concentration-compactness principles which have been proved to be very effective in variational problems involving critical

Sobolev growth conditions. These principles are currently named as the first and second concentration-compactness principles (CCP1, CCP2). The proof of these concentration-compactness principles can also be found in [52, 53]. In [11], Chabrowski formulated a variant of these two principles, namely, the concentration-compactness principle at infinity (CCP$_\infty$) for both critical and subcritical cases. As to p-Laplacian problems with critical growth conditions, there are many results (see [1, 7-10, 15-18, 26-28, 35, 43-47] and the references therein). But results on the $p(x)$-Laplacian problems with critical growth conditions are rare.

In [24], Fu gave the concentration-compactness principle in $L^{p(\cdot)}(\Omega)$ space, and discussed the existence of at least one nontrivial solution. Our aim here is to deal with the existence of multiple solutions for $p(x)$-Laplacian problems involving critical growth conditions. We obtain the existence of infinitely many pairs of solutions by the Fountain Theorem and the Dual Fountain Theorem. Especially, we give a variant of concentration-compactness principle. These results are extension of results of p-Laplacian problems.

This paper is organized as follows. In Section 2, we introduce some basic properties of the variable exponent Sobolev spaces, and also present a variant of concentration-compactness principle. In Section 3, several important properties of $p(x)$-Laplacian are presented. Finally, we give the main results and the proofs in Section 4.

2. Weighted Variable Exponent Lebesgue and Sobolev Spaces

In order to discuss the problem (P), we need the functional space $W^{1,p(\cdot)}(\Omega)$ which is called variable exponent Sobolev space. To deal with critical nonlinearities, we also need a variant of the concentration-compactness principle. Let $S(\Omega)$ be the set of all measurable real valued functions defined on Ω. Denote

$$h^+ = ess\sup_{x\in\overline{\Omega}} h(x),\ h^- = ess\inf_{x\in\overline{\Omega}} h(x), \text{ for any } h\in S(\overline{\Omega}),$$

$$C_+(\overline{\Omega}) = \left\{ h \,\middle|\, h\in C(\overline{\Omega}),\ h^-\geq 1 \text{ for } x\in\overline{\Omega} \right\},$$

$$L^{p(\cdot)}(\Omega) = \left\{ u \mid u\in S(\Omega),\ \textstyle\int_\Omega |u(x)|^{p(x)}\,dx < \infty \right\}.$$

The Luxemburg norm on $L^{p(\cdot)}(\Omega)$ is defined by

$$|u|_{p(\cdot)} = \inf\left\{ \lambda>0 \,\middle|\, \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$

Under the norm as above $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ becomes a Banach space, which is called variable exponent Lebesgue space.

The space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u\in L^{p(\cdot)}(\Omega) \,\middle|\, |\nabla u|\in L^{p(\cdot)}(\Omega) \right\},$$

in which the norm is defined by

$$\|u\|_{p(\cdot)} = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}, \forall u \in W^{1,p(\cdot)}(\Omega).$$

Let $r^0(\cdot)$ be the conjugate function of $r(\cdot)$, namely $r^0(x) = \begin{cases} \frac{r(x)}{r(x)-1}, r \in C(\overline{\Omega}) \\ 1, r = \infty \end{cases}$.

Proposition 2.1 . (see [19]).

(*i*) *If* $q \in L^\infty(\Omega)$, $1 < q^- \le q^+ < \infty$, *then the space* $(L^{q(\cdot)}(\Omega), |\cdot|_{q(\cdot)})$ *is a separable, uniformly convex Banach space, and it's conjugate space is* $L^{q^0(\cdot)}(\Omega)$ *where* $\frac{1}{q(x)} + \frac{1}{q^0(x)} \equiv 1$. *For any* $u \in L^{q(\cdot)}(\Omega)$ *and* $v \in L^{q^0(\cdot)}(\Omega)$, *we have*

$$\left| \int_\Omega uv dx \right| \le (\frac{1}{q^-} + \frac{1}{(q^0)^-}) |u|_{q(\cdot)} |v|_{q^0(\cdot)}.$$

(*ii*) *If* $\Omega \subset \mathbb{R}^N$ *is open bounded,* $1 \le p_1, p_2 \in C(\overline{\Omega})$, $p_1(x) \le p_2(x)$ *for any* $x \in \Omega$, *then* $L^{p_2(\cdot)}(\Omega) \subset L^{p_1(\cdot)}(\Omega)$, *and the imbedding is continuous.*

Proposition 2.2. (see [19]). *If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)} dx, \forall u \in L^{p(\cdot)}(\Omega),$$

then

(*i*) $|u|_{p(\cdot)} < 1(=1; > 1) \iff \rho(u) < 1(=1; > 1)$;

(*ii*) $|u|_{p(\cdot)} > 1 \Longrightarrow |u|_{p(\cdot)}^{p^-} \le \rho(u) \le |u|_{p(\cdot)}^{p^+}$; $|u|_{p(\cdot)} < 1 \Longrightarrow |u|_{p(\cdot)}^{p^-} \ge \rho(u) \ge |u|_{p(\cdot)}^{p^+}$;

(*iii*) $|u|_{p(\cdot)} \to 0 \iff \rho(u) \to 0$; $|u|_{p(\cdot)} \to \infty \iff \rho(u) \to \infty$.

Proposition 2.3. (see [19]). *If* u, $u_n \in L^{p(\cdot)}(\Omega)$, $n = 1, 2, \cdots$, *then the following statements are equivalent to each other:*

(1) $\lim_{k\to\infty} |u_k - u|_{p(\cdot)} = 0$;

(2) $\lim_{k\to\infty} \rho(u_k - u) = 0$;

(3) $u_k \to u$ *in measure in* Ω *and* $\lim_{k\to\infty} \rho(u_k) = \rho(u)$.

Proposition 2.4. (see [19]).

(*i*) $W^{1,p(\cdot)}(\Omega)$ *is a separable reflexive Banach space;*

(*ii*) *If* $1 \le q \in C(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then the imbedding from* $W^{1,p(\cdot)}(\Omega)$ *to* $L^{q(\cdot)}(\Omega)$ *is compact;*

(*iii*) *If* p *is Lipschitz continuous,* q *is measurable and satisfies* $1 \le q(x) \le p^*(x)$ *for any* $x \in \overline{\Omega}$, *then the imbedding from* $W^{1,p(\cdot)}(\Omega)$ *to* $L^{q(\cdot)}(\Omega)$ *is continuous.*

Proposition 2.5. (see [19]). *If Ω is an open subset of $\mathbb{R}^N$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies*

$$|f(x,s)| \leq a(x) + b\,|s|^{p_1(x)/p_2(x)} \text{ for any } x \in \Omega, s \in \mathbb{R},$$

where $1 \leq p_1$, $p_2 \in C(\Omega)$, $a(\cdot) \in L^{p_2(\cdot)}(\Omega)$, $a(x) \geq 0$, $b \geq 0$ is a constant, then the Nemytsky operator from $L^{p_1(\cdot)}(\Omega)$ to $L^{p_2(\cdot)}(\Omega)$ defined by $(N_f u)(x) = f(x, u(x))$ is a continuous and bounded operator.

Denote $\|u\|'_{p(\cdot)} = \inf\{\lambda > 0 \mid \int_\Omega \left|\frac{\nabla u}{\lambda}\right|^{p(x)} dx + \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1\}$, then it is easy to see that $\|\cdot\|'_{p(\cdot)}$ is an equivalence norm of $\|\cdot\|_{p(\cdot)}$ on $W^{1,p(\cdot)}(\Omega)$. In the following, we will use $\|\cdot\|'_{p(\cdot)}$ instead of $\|\cdot\|_{p(\cdot)}$ on $W^{1,p(\cdot)}(\Omega)$.

Let $\mathbb{M}(\overline{\Omega})$ denote the class of nonnegative Borel measures of finite total mass, and $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$ in $\mathbb{M}(\overline{\Omega})$ is defined by $\int_{\overline{\Omega}} \eta d\mu_\varepsilon \to \int_{\overline{\Omega}} \eta d\mu$ for every test function $\eta \in C(\overline{\Omega})$.

Proposition 2.6. (see [24]). *Assume Ω is an open bounded domain in $\mathbb{R}^N$, p is Lipschitz continuous on $\overline{\Omega}$ and satisfy $1 < p(x) < N$. Let $\{\omega_\varepsilon\}$ be a sequence in $W_0^{1,p(\cdot)}(\Omega)$ of norm $\|\nabla \omega_\varepsilon\|_{p(\cdot)} \leq 1$ such that*

$$\omega_\varepsilon \rightharpoonup \omega \text{ in } W_0^{1,p(\cdot)}(\Omega),\ |\nabla \omega_\varepsilon|^{p(x)} \overset{*}{\rightharpoonup} \mu \text{ in } \mathbb{M}(\overline{\Omega}),\ |\omega_\varepsilon|^{p^*(x)} \overset{*}{\rightharpoonup} \upsilon \text{ in } \mathbb{M}(\overline{\Omega}).$$

Denote

$$C^* = \sup\{\int_\Omega |\omega_\varepsilon|^{p^*(x)} dx \,\Big|\, \omega_\varepsilon \in W_0^{1,p(\cdot)}(\Omega), |\nabla \omega_\varepsilon|_{p(\cdot)} \leq 1\}$$

and then $0 < C^* < +\infty$. The limit measures are of the form

$$\mu = |\nabla \omega|^{p(x)} + \sum_{j \in J} \mu_j \delta_{x_j} + \widetilde{\mu}, \mu(\overline{\Omega}) \leq 1,\ \upsilon = |\omega|^{p^*(x)} + \sum_{j \in J} \upsilon_j \delta_{x_j}, \upsilon(\overline{\Omega}) \leq C^*,$$

where $x_j \in \overline{\Omega}$, J is a countable set, $\widetilde{\mu} \in \mathbb{M}(\overline{\Omega})$ is a nonatomic positive measure. The atoms and the regular part satisfy the generalized Sobolev inequality

$$\upsilon(\overline{\Omega}) \leq C^* \max\{\mu(\overline{\Omega})^{\frac{p^{*+}}{p^-}}, \mu(\overline{\Omega})^{\frac{p^{*-}}{p^+}}\},\ \upsilon_j \leq C^* \max\{\mu_j^{\frac{p^{*+}}{p^-}}, \mu_j^{\frac{p^{*-}}{p^+}}\}.$$

We have the following version of the concentration-compactness principle:

Theorem 2.7. *Assume Ω is an open bounded domain in $\mathbb{R}^N$, p is Lipschitz continuous on $\overline{\Omega}$ satisfying $1 < p(x) < N$, $1 < q(x) \leq p^*(x)$, $\varepsilon_n \to 0^+$ as $n \to +\infty$. Let $\{\omega_n\}$ be a sequence in $W_0^{1,p(\cdot)}(\Omega)$ of norm $\|\nabla \omega_n\|_{p(\cdot)} \leq 1$ such that*

$$\omega_n \rightharpoonup \omega \text{ in } W_0^{1,p(\cdot)}(\Omega),\ |\nabla \omega_n|^{p(x)} \overset{*}{\rightharpoonup} \mu \text{ in } \mathbb{M}(\overline{\Omega}),\ |\omega_n|^{q(x)-\varepsilon_n} \overset{*}{\rightharpoonup} \upsilon^{\#} \text{ in } \mathbb{M}(\overline{\Omega}).$$

Denote

$$C_q^* = \sup\{|\omega|_{q(\cdot)}^{q^+} + 1 \,\Big|\omega \in W_0^{1,p(\cdot)}(\Omega), |\nabla\omega_\varepsilon|_{p(\cdot)} \le 1\},$$

and then $0 < C_q^* < +\infty$. The limit measures are of the form

$$\begin{aligned}\mu &= |\nabla\omega|^{p(x)} + \sum_{j\in J}\mu_j\delta_{x_j} + \widetilde{\mu}, \mu(\overline{\Omega}) \le 1,\\ \upsilon^{\#} &= |\omega|^{q(x)} + \sum_{j\in J}\upsilon_j^{\#}\delta_{x_j}, \upsilon^{\#}(\overline{\Omega}) \le C_q^* + 1,\\ \upsilon^{\#}(\{x\}) &\le \upsilon(\{x\}), \forall x \in \Omega,\end{aligned}$$

where $x_j \in \overline{\Omega}$, J is a countable set, and $\widetilde{\mu} \in \mathbb{M}(\overline{\Omega})$ is a non-atomic positive measure. The atoms and the regular part satisfy the generalized Sobolev inequality

$$\upsilon^{\#}(\overline{\Omega}) \le C_q^* \max\{\mu(\overline{\Omega})^{\frac{q^+}{p^-}}, \mu(\overline{\Omega})^{\frac{q^-}{p^+}}\},\ \upsilon_j^{\#} \le C_q^* \max\{\mu_j^{\frac{q^+}{p^-}}, \mu_j^{\frac{q^-}{p^+}}\}.$$

In order to prove the Theorem 2.7, we need the following Lemma.

Lemma 2.8. *Assume Ω is an open bounded domain in $\mathbb{R}^N$, $\{f_n\}$ is bounded in $L^{p(\cdot)}(\Omega)$ and $f_n \to f \in L^{p(\cdot)}(\Omega)$ a.e. on Ω. If $1 < p(x) < N$, and $\varepsilon_n \to 0^+$ as $n \to +\infty$, then*

$$\lim_{n\to\infty}\{\int_\Omega |f_n|^{p(x)-\varepsilon_n}\,dx - \int_\Omega |f_n - f|^{p(x)-\varepsilon_n}\,dx\} = \int_\Omega |f|^{p(x)}\,dx.$$

Proof. Without loss of generality, we may assume that $\varepsilon_n \le \frac{1}{2}, n = 1, 2, \cdots$. It is easy to see that

$$\lim_{n\to\infty}\int_\Omega |f|^{p(x)-\varepsilon_n}\,dx = \int_\Omega |f|^{p(x)}\,dx.$$

Now it suffices to show that

$$\lim_{n\to\infty}\{\int_\Omega |f_n|^{p(x)-\varepsilon_n}\,dx - \int_\Omega |f_n - f|^{p(x)-\varepsilon_n}\,dx - \int_\Omega |f|^{p(x)-\varepsilon_n}\,dx\} = 0. \tag{1}$$

Denote

$$W_{\varepsilon,n}(x) = \left[\left||f_n|^{p(x)-\varepsilon_n} - |f_n - f|^{p(x)-\varepsilon_n} - |f|^{p(x)-\varepsilon_n}\right| - \varepsilon\,|f_n - f|^{p(x)-\varepsilon_n}\right]_+,$$

where $[a]_+ = \max\{a, 0\}$. Obviously, $W_{\varepsilon,n}(x) \to 0$ a.e. on Ω, as $n \to \infty$.

Similarly to the proof of Lemma 2.1 of [24], we have

$$\left||f_n|^{p(x)-\varepsilon_n} - |f_n - f|^{p(x)-\varepsilon_n} - |f|^{p(x)-\varepsilon_n}\right| \le \varepsilon\,|f_n - f|^{p(x)-\varepsilon_n} + C(\varepsilon)\,|f|^{p(x)-\varepsilon_n},$$

where ε and $C(\varepsilon)$ are independent of n.

Therefore, $W_{\varepsilon,n}(x) \leq C(\varepsilon)\,|f|^{p(x)-\varepsilon_n} \leq C(\varepsilon)(|f|^{p(x)}+1) \in L^1(\Omega)$. By Lebesgue's Dominated Convergence Theorem, we have $\int_\Omega W_{\varepsilon,n}(x)dx \to 0$, as $n \to \infty$. Similarly to the proof of Lemma 2.1 of [24], we can see that (1) holds. ■

Proof of Theorem 2.7. There exists a subsequence of $\{u_n\}$(for simplicity we still denote it as $\{u_n\}$) such that $u_n(x) \to u(x)$ a.e. on Ω. By Lemma 2.8, we have

$$\liminf_{n\to\infty} \int_{\Omega'} |u_n|^{q(x)-\varepsilon_n}\,dx - \int_{\Omega'} |u|^{q(x)}\,dx = \liminf_{n\to\infty} \int_{\Omega'} |u_n - u|^{q(x)-\varepsilon_n}\,dx,$$

$$\liminf_{n\to\infty} \int_{\Omega} |u_n|^{q(x)-\varepsilon_n}\,\eta dx - \int_{\Omega} |u|^{q(x)}\,\eta dx = \liminf_{n\to\infty} \int_{\Omega} |u_n - u|^{q(x)-\varepsilon_n}\,\eta dx,$$

for every sub-domain $\Omega' \subset \Omega$ and $\eta \in C(\overline{\Omega})$. Thus

$$\overline{\upsilon}^{\#} = \upsilon^{\#} - |u|^{q(x)} = \sum_{j\in J} \upsilon_j^{\#}\delta_{x_j} + \widetilde{\upsilon}^{\#},$$

with non-atomics $\widetilde{\upsilon}^{\#} \in \mathbb{M}(\overline{\Omega})$. Similarly to the proof of Theorem 3.1 of [24], we can see that $\widetilde{\upsilon}^{\#} = 0$. Thus $\upsilon^{\#} = |\omega|^{q(x)} + \sum_{j\in J} \upsilon_j^{\#}\delta_{x_j}$.

For any $x_0 \in \overline{\Omega}$ and $\forall \varepsilon > 0$, let $\phi_\varepsilon \in C(\mathbb{R}^N)$ with $\phi_\varepsilon(x) \geq 0, \phi_\varepsilon(x_0) = 1$, $\phi_\varepsilon(x) = 0$ when $|x - x_0| \geq \varepsilon$. We have

$$\begin{aligned} \upsilon^{\#}(\{x_0\}) &\leq \int_\Omega \phi_\varepsilon(x)\upsilon^{\#}dx = \lim_{n\to\infty} \int_\Omega \phi_\varepsilon(x)\,|u_n|^{q(x)-\varepsilon_n}\,dx \\ &\leq \lim_{n\to\infty} \int_\Omega \phi_\varepsilon(x)\{|u_n|^{q(x)} + 1\}dx \leq \int_\Omega \phi_\varepsilon(x)\upsilon dx + C\varepsilon^N. \end{aligned}$$

Letting $\varepsilon \to 0$, we have $\upsilon^{\#}(\{x_0\}) \leq \upsilon(\{x_0\})$. ■

3. Properties of Operators and Variational Principle

In the following, we will discuss the properties of the $p(x)$-Laplacian operator and Nemytsky operator. Also, we will present several variational principles.

From now on, the letters c, c_i, C, C_i, $i = 1, 2, ...$, denote positive constants which may vary from line to line but are independent of the terms which take part in any limit process. Denote $X := W_0^{1,p(\cdot)}(\Omega)$. We Consider the following functional

$$J(u) = \int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}\,|u|^{p(x)}\,dx,\; u \in X.$$

Obviously (see [12]), $J \in C^1(X, \mathbb{R})$ and it is weak lower semi-continuous. Denote $L = J' : X \to X^*$, then we have

$$(L(u), v) = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_\Omega |u|^{p(x)-2} uv dx, \forall u, v \in X.$$

Proposition 3.1. (see [21]).

(i) *$L : X \to X^*$ is continuous, bounded and strictly monotone;*

(ii) *L is a mapping of type (S_+), i.e. if $u_n \rightharpoonup u$ in X and $\overline{\lim_{n\to\infty}} (L(u_n) - L(u), u_n - u) \le 0$, then $u_n \to u$ in X;*

(iii) *$L : X \to X^*$ is a homeomorphism.*

Denote $F(x, u) = \int_0^u f(x, t)dt$, $\Psi(u) = \int_\Omega F(x, u)dx$, $(\Psi'(u), v) = \int_\Omega f(x, u)vdx$. The corresponding functional of (P) is

$$\varphi(u) = J(u) - \Psi(u), \forall u \in X.$$

Then we have the following theorem.

Theorem 3.2.

(i) *$\Psi \in C^1(X, \mathbb{R})$;*

(ii) *If $f(x, u)/ |u|^{p^*(x)-1} \to 0$ as $|u| \to \infty$, then Ψ and Ψ' are weak-strong continuous, i.e., $u_n \rightharpoonup u$ implies $\Psi(u_n) \to \Psi(u)$ and $\Psi'(u_n) \to \Psi'(u)$.*

Proof.

(i) From the continuity of the Nemytsky operator, we can see that both Ψ and Ψ' are continuous.

(ii) Since $u_n \rightharpoonup u$, we have $|u_n - u|_{p(\cdot)} \to 0$ and $u_n \to u$ a.e. on $\overline{\Omega}$. Thus, $F(x, u_n(x)) \to F(x, u(x))$ a.e. on $\overline{\Omega}$. Clearly,

$$\int_U |F(x, u_n)| dx \le \int_U [\varepsilon |u_n|^{p^*(x)} + C(\varepsilon)]dx, \forall U \subset \Omega,$$

then $\{|F(x, u_n)|\}$ is uniformly integrable, and then$\{|F(x, u_n) - F(x, u)|\}$ is uniformly integrable. Noticing the boundedness of the domain Ω, we have

$$\lim_{n\to\infty} \int_\Omega |F(x, u_n) - F(x, u)| dx = \int_\Omega \lim_{n\to\infty} |F(x, u_n) - F(x, u)| dx = 0.$$

Similarly, we can get the weak-strong continuity of Ψ'. ∎

Since X is a reflexive and separable Banach space, there are sequences $\{e_j\} \subset X$ and $\left\{e_j^*\right\} \subset X^*$ such that

$$X = \overline{span}\{e_j, j = 1, 2, \cdots\}, \quad X^* = \overline{span}^{w^*}\{e_j^*, j = 1, 2, \cdots\},$$

and $< e_j^*, e_i >= \begin{cases} 1, i = j, \\ 0, i \neq j. \end{cases}$

For convenience, we write

(2) $$X_j = span\{e_j\},\, Y_k = \overset{k}{\underset{j=1}{\oplus}} X_j,\, Z_k = \overline{\overset{\infty}{\underset{j=k}{\oplus}} X_j}.$$

Definition 3.3. (i) We say φ satisfies (PS) condition in X, if any sequence $\{u_n\} \subset X$ such that $\{\varphi(u_n)\}$ is bounded and $\|\varphi'(u_n)\|_{X^*} \to 0$ as $n \to \infty$, has a convergent subsequence; (ii) We say φ satisfies $(PS)_c^*$ condition in X, if any sequence $\{u_{n_j}\}\subset X$ such that $n_j \to \infty$, $u_{n_j} \in Y_{n_j}$, $\varphi(u_{n_j}) \to c$ and $(\varphi|_{Y_{n_j}})'(u_{n_j}) \to 0$, contains a subsequence converging to a critical point of φ.

Let $f(x,t) = g(x,t) + h(x,t)$. Denote

$$G(x,t) = \int_0^t g(x,s)ds, H(x,t) = \int_0^t h(x,s)ds.$$

We assume

(B$_1$). There exist a positive constant M and a function $\theta(\cdot) \in C^1(\overline{\Omega})$ satisfying

$$p(x) < \theta(x) \leq p^*(x), \forall x \in \overline{\Omega},$$

such that h satisfies

$$0 < H(x,s) \leq \frac{s}{\theta(x)} h(x,s), \forall x \in \overline{\Omega}, |s| \geq M.$$

(B$_2$). For the function $\theta(\cdot)$ in **(B$_1$)**, there exists a small positive constant δ such that g satisfies

$$|g(x,s)| \leq |s|^{\frac{\theta(x)}{1+\delta}-1}, \forall x \in \overline{\Omega}, |s| \geq M.$$

Lemma 3.4. *If (B_1) and (B_2) are satisfied, then every (PS) sequence of φ in X is bounded.*

Proof.. The conditions **(B$_1$)** and **(B$_2$)** together imply that

$$H(x,s) \geq |s|^{\theta(x)}, \forall x \in \overline{\Omega}, \text{ when } |s| \text{ is large enough},$$

$$|G(x,s)| + |sg(x,s)| \leq (1+\theta(x))\, |s|^{\frac{\theta(x)}{1+\delta}} + C, \forall (x,s) \in \overline{\Omega} \times \mathbb{R}.$$

Denote

$$l_1 = \min_{x\in\overline{\Omega}} \left(\frac{1}{p(x)} - \frac{1+\delta}{\theta(x)}\right),$$

where the positive constant δ is small enough such that $l_1 > 0$.

Let $\{u_n\}$ be a (PS) sequence. Since $\theta \in C^1(\overline{\Omega})$, we have

$$
\begin{aligned}
& c+\|u_n\|_{p(\cdot)} \\
\geq\ & \varphi(u_n)-(\varphi'(u_n),\frac{1+\delta}{\theta(x)}u_n) \\
=\ & \int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)}+|u_n|^{p(x)})dx-\int_\Omega F(x,u_n)dx \\
& -\int_\Omega \frac{1+\delta}{\theta(x)}(|\nabla u_n|^{p(x)}+|u_n|^{p(x)})dx \\
& +\int_\Omega \frac{1+\delta}{\theta(x)}u_n f(x,u_n)dx+\int_\Omega \frac{1+\delta}{\theta^2(x)}u_n\left|\nabla u_n\right|^{p(x)-2}\nabla u_n\nabla\theta(x)dx \\
\geq\ & \int_\Omega (\frac{1}{p(x)}-\frac{1+\delta}{\theta(x)})(|\nabla u_n|^{p(x)}+|u_n|^{p(x)})dx+\int_\Omega \frac{\delta}{\theta(x)}H(x,u_n)dx \\
& -\int_\Omega \frac{(1+\delta)\left|\nabla\theta(x)\right|}{\theta^2(x)}\left|u_n\right|\left|\nabla u_n\right|^{p(x)-1}dx-\int_\Omega C_1\left|u_n\right|^{\frac{\theta(x)}{1+\delta}}dx-C_2 \\
\geq\ & l_1\int_\Omega (|\nabla u_n|^{p(x)}+|u_n|^{p(x)})dx+\int_\Omega \frac{\delta}{\theta(x)}H(x,u_n)dx \\
& -\int_\Omega \frac{(1+\delta)\left|\nabla\theta(x)\right|}{\theta^2(x)}\left|u_n\right|\left|\nabla u_n\right|^{p(x)-1}dx-\int_\Omega C_1\left|u_n\right|^{\frac{\theta(x)}{1+\delta}}dx-C_2.
\end{aligned}
$$

Since $\theta\in C(\overline{\Omega})$ it follows that

$$
\begin{aligned}
& \frac{(1+\delta)\left|\nabla\theta(x)\right|}{\theta^2(x)}\left|u_n\right|\left|\nabla u_n\right|^{p(x)-1} \\
\leq\ & C_4\frac{1}{p(x)}(\frac{1}{\varepsilon_1}\left|u_n\right|)^{p(x)}+C_4\frac{p(x)-1}{p(x)}(\varepsilon_1\left|\nabla u_n\right|^{p(x)-1})^{\frac{p(x)}{p(x)-1}} \\
=\ & C_4\frac{1}{p(x)}\frac{1}{\varepsilon_1^{p(x)}}\left|u_n\right|^{p(x)}+C_4\frac{p(x)-1}{p(x)}\varepsilon_1^{\frac{p(x)}{p(x)-1}}\left|\nabla u_n\right|^{p(x)}.
\end{aligned}
$$

It is not hard to see that

$$
\begin{aligned}
\int_\Omega C_1\left|u_n\right|^{\frac{\theta(x)}{1+\delta}}dx &\leq \int_\Omega \{\frac{\delta}{1+\delta}(\frac{C_1}{\varepsilon_1})^{\frac{1+\delta}{\delta}}+\frac{1}{1+\delta}(\varepsilon_1\left|u_n\right|^{\frac{\theta(x)}{1+\delta}})^{1+\delta}\}dx \\
&= C_5+\int_\Omega \frac{\varepsilon_1}{1+\delta}\left|u_n\right|^{\theta(x)}dx.
\end{aligned}
$$

When the positive constants ε_1 is small enough, we have

$$
\begin{aligned}
& c+\|u_n\|_{p(\cdot)}\geq\varphi(u_n)-(\varphi'(u_n),\frac{1+\delta}{\theta(x)}u_n) \\
\geq\ & \frac{2l_1}{3}\int_\Omega (|\nabla u_n|^{p(x)}+|u_n|^{p(x)})dx-C_6.
\end{aligned}
$$

Thus $\{\|u_n\|_{p(\cdot)}\}$ is bounded. ■

Lemma 3.5. *If* $(\boldsymbol{B}_1)$ *and* $(\boldsymbol{B}_2)$ *are satisfied,* $\{u_n\}$ *is a bounded* (PS) *sequence of* φ*, then there exists a small enough positive constant* C_0 *such that, if* f *satisfies*

$$|f(x,s)| \leq C + C_0 |s|^{p^*(x)-1}, \ \forall x \in \overline{\Omega},$$

then $\{u_n\}$ *has a convergent subsequence in* X.

Proof. Let $\{u_n\}$ be a (PS) sequence of φ, i.e.

$$\varphi(u_n) \to c, \varphi'(u_n) \to 0 \text{ as } n \longrightarrow \infty.$$

Since $\{u_n\}$ is bounded, there exists a $u \in X$, such that $u_n \rightharpoonup u$ in X. By Proposition 2.6, we may assume that there exist $\mu, \nu \in \mathbb{M}(\overline{\Omega})$ and sequence $\{x_j\}_{j\in J}$ in $\overline{\Omega}$ such that

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W_0^{1,p(\cdot)}(\Omega), \\ |\nabla u_n|^{p(x)} \overset{*}{\rightharpoonup} \mu &= |\nabla u|^{p(x)} + \sum_{j\in J} \mu_j \delta_{x_j} + \widetilde{\mu}, \text{ in } \mathbb{M}(\overline{\Omega}), \\ |u_n|^{p^*(x)} \overset{*}{\rightharpoonup} \upsilon &= |u|^{p^*(x)} + \sum_{j\in J} \upsilon_j \delta_{x_j}, \text{ in } \mathbb{M}(\overline{\Omega}), \\ \upsilon_j &\leq C_{p^*}^* \max\{\mu_j^{\frac{p^{+*}}{p^-}}, \mu_j^{\frac{p^{-*}}{p^+}}\}, \end{aligned}$$

where

$$C_{p^*}^* = \sup\{|\omega|_{p^*(\cdot)}^{p^{*+}} + 1 \,\Big|\, \omega \in W_0^{1,p(x)}(\Omega), |\nabla\omega|_{p(\cdot)} \leq 1\}$$

and $0 < C_{p^*}^* < +\infty$.

Next we will complete the proof of this Theorem in three steps.

Step 1. We claim that $\mu(\{x_j\}) = \upsilon(\{x_j\}) = 0$ for all $j = 1, 2, \cdots$.

Obviously, there exists $r_n > 0$ such that

$$\begin{aligned} p^-(x_n) \ : \ = \inf_{y\in B_r(x_n)\cap\overline{\Omega}} p(y) \leq p^+(x_n) := \sup_{y\in B_r(x_n)\cap\overline{\Omega}} p(y) \\ < p^{*-}(x_n) := \inf_{y\in B_r(x_n)\cap\overline{\Omega}} p^*(y) \leq p^{*+}(x_n) := \sup_{y\in B_r(x_n)\cap\overline{\Omega}} p^*(y), \forall r \in (0, r_n]. \end{aligned}$$

For every $\varepsilon > 0$, we set $\phi_\varepsilon(x) = \phi(\frac{x-x_1}{\varepsilon}), x \in \Omega$, where $\phi \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $B_1\{0\}$ and $\phi \equiv 0$ in $\mathbb{R}^N \backslash B_2\{0\}$ and $|\nabla\phi| \leq 2$. Noting that $\varphi'(u_n) \to 0$ in X^* as $n \longrightarrow \infty$ and $\{u_n\}$ is bounded, we have

$$\begin{aligned}&\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(\phi_\varepsilon u_n)dx+\int_\Omega |u_n|^{p(x)-2}u_n\phi_\varepsilon u_n dx\\ =&\int_\Omega f(x,u_n)\phi_\varepsilon u_n dx+o(1)\\ \le&\int_\Omega (C|u_n|+C_0|u_n|^{p^*(x)})\phi_\varepsilon dx+o(1).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_\Omega \phi_\varepsilon |\nabla u_n|^{p(x)}dx+\int_\Omega |u_n|^{p(x)}\phi_\varepsilon dx+\int_\Omega u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi_\varepsilon dx\\ \le&\int_\Omega (C|u_n|+C_0|u_n|^{p^*(x)})\phi_\varepsilon dx+o(1).\end{aligned}\tag{3}$$

Due to the bounded-ness of $\{u_n\}$ in X, we may assume

$$|\nabla u_n|^{p(x)-2}\nabla u_n\rightharpoonup T\in(L^{p^0(\cdot)}(\Omega))^N,\ f(x,u_n)\rightharpoonup g(x)\in L^{(p^*(\cdot))^0}(\Omega).$$

Noting $\varphi'(u_n)\to 0$ in X^* as $n\longrightarrow\infty$, we also have

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla(\phi_\varepsilon u)dx+\int_\Omega |u_n|^{p(x)-2}u_n\phi_\varepsilon u dx=\int_\Omega f(x,u_n)\phi_\varepsilon u dx+o(1).$$

Thus,

$$\int_\Omega T\cdot\nabla(\phi_\varepsilon u)dx+\int_\Omega |u|^{p(x)}\phi_\varepsilon dx=\int_\Omega f(x,u)u\phi_\varepsilon dx.\tag{4}$$

We claim

$$\int_\Omega u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi_\varepsilon dx\to\int_\Omega uT\nabla\phi_\varepsilon dx \text{ as } n\to\infty.\tag{5}$$

In fact,

$$\begin{aligned}&\int_\Omega \{u_n|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi_\varepsilon-uT\nabla\phi_\varepsilon\}dx\\ =&\int_\Omega (u_n-u)|\nabla u_n|^{p(x)-2}\nabla u_n\nabla\phi_\varepsilon dx\\ &+\int_\Omega u\nabla\phi_\varepsilon\{|\nabla u_n|^{p(x)-2}\nabla u_n-T\}dx\to 0 \text{ as } n\to\infty.\end{aligned}$$

From (3), (4) and (5) it follows that

$$
\begin{aligned}
&\int_\Omega \phi_\varepsilon d\mu + \int_\Omega |u|^{p(x)}\, \phi_\varepsilon dx \\
\leq\ & \int_\Omega C\,|u|\, \phi_\varepsilon dx + \int_\Omega C_0 \phi_\varepsilon d\upsilon - \int_\Omega uT\nabla\phi_\varepsilon dx \\
=\ & \int_\Omega C\,|u|\, \phi_\varepsilon dx + \int_\Omega C_0 \phi_\varepsilon d\upsilon \\
&-\{\int_\Omega f(x,u)u\phi_\varepsilon dx - \int_\Omega |u|^{p(x)}\, \phi_\varepsilon dx - \int_\Omega \phi_\varepsilon T\cdot\nabla u dx\}.
\end{aligned}
$$

Letting $\varepsilon \to 0$, we get $\mu(\{x_1\}) \leq C_0\upsilon(\{x_1\})$ or $\mu_1 \leq C_0\upsilon_1$.

Similarly, $\mu(\{x_j\}) \leq C_0\upsilon(\{x_j\})$ or $\mu_j \leq C_0\upsilon_j, j = 2, 3, \cdots$.

Suppose that $\mu(\{x_j\}) > 0$ for some j. Since $\{u_n\}$ is bounded in $W_0^{1,p(\cdot)}(\Omega)$, there is a constant M_* such that $\int_\Omega |u_n|^{p^*(x)}\, dx \leq M_* < 0$ for all n. If $\mu(\{x_j\}) \geq 1$, then $\upsilon_j \geq \frac{1}{C_0}\left|\frac{\upsilon_j}{C^*_{p^*}}\right|^{\frac{p^-(x_j)}{p^{*+}(x_j)}}$. It can be rewrited as

$$
\upsilon_j \geq \left|C_0(C^*_{p^*})^{\frac{p^-(x_j)}{p^{*+}(x_j)}}\right|^{\frac{-1}{1-\frac{p^-(x_j)}{p^{*+}(x_j)}}}.
$$

Similarly, if $\mu(\{x_j\}) < 1$ then $\upsilon_j \geq \frac{1}{C_0}\left|\frac{\upsilon_j}{C^*_{p^*}}\right|^{\frac{p^+(x_j)}{p^{*-}(x_j)}}$ and

$$
\upsilon_j \geq \left|C_0(C^*_{p^*})^{\frac{p^+(x_j)}{p^{*-}(x_j)}}\right|^{\frac{-1}{1-\frac{p^+(x_j)}{p^{*-}(x_j)}}}.
$$

Due to the definition of M_*, we also have

$$
\sum_{\mu(\{x_j\})\geq 1} \upsilon_j + \sum_{\mu(\{x_j\})<1} \upsilon_j \leq M_*.
$$

Noting that M_* is a constant which is only dependent on $\{u_n\}$. When C_0 (depending on M_*) is small enough, we reach a contradiction. Step 1 is completed.

Step 2. We claim that $u_n \to u$ strongly in $L^{p^*(x)}(\Omega)$ as $n \to \infty$.

Since $|u_n|^{p^*(x)} \overset{*}{\rightharpoonup} \nu = |u|^{p^*(x)}$, we have

$$
\lim_{n\to\infty} \int_\Omega |u_n|^{p^*(x)}\, dx = \int_\Omega |u|^{p^*(x)}\, dx,
$$

which together with $|u_n|^{p^*(x)} \to |u|^{p^*(x)}$ in measure implies that $\{|u_n|^{p^*(x)}\}$ is uniformly integrable.

Obviously,

$$|u_n - u|^{p^*(x)} \leq 2^{p^*(x)}(|u_n|^{p^*(x)} + |u|^{p^*(x)}).$$

Thus $\{|u_n - u|^{p^*(x)}\}$ is uniformly integrable. Therefore

$$\lim_{n\to\infty} \int_\Omega |u_n - u|^{p^*(x)}\, dx = \int_\Omega \lim_{n\to\infty} |u_n - u|^{p^*(x)}\, dx = 0.$$

Step 3. We claim that $u_n \to u$ strongly in X as $n \to \infty$.

Since $\varphi'(u_n) = J'(u_n) - \Psi'(u_n) \to 0$ and $u_n \to u$ strong in $L^{p^*(x)}(\Omega)$ as $n \to \infty$, we have $\Psi'(u_n) \to \Psi'(u)$ and $J'(u_n) \to \Psi'(u)$ as $n \to \infty$. As $L = J'$ is a homeomorphism, we have $u_n \to L^{-1}(\Psi'(u))$ in X as $n \to \infty$. ■

Lemma 3.6. *Assume $\Theta : X \to \mathbb{R}$ is weakly-strongly continuous and $\Theta(0) = 0$, $\gamma > 0$ is a fixed number. Let*

$$\beta_k = \beta_k(\gamma) = \sup\{\Theta(u) \mid \|u\| \leq \gamma, u \in Z_k\}, \tag{6}$$

then $\beta_k \to 0$ as $k \to \infty$.

Lemma 3.7. (see [23]). *If $|u(\cdot)|^{q(\cdot)} \in L^{s(\cdot)/q(\cdot)}(\Omega)$, where $s(\cdot), q(\cdot) \in L^\infty_+(\Omega)$, $q(x) \leq s(x)$, then $u \in L^{s(\cdot)}(\Omega)$ and there is a number $\overline{q} \in [q^-, q^+]$ such that $\left||u(\cdot)|^{q(\cdot)}\right|_{s(\cdot)/q(\cdot)} = (|u|_{s(\cdot)})^{\overline{q}}$.*

Proposition 3.8. (Fountain theorem, see [52, 53]). *Assume X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$ is an even functional and satisfies (PS) condition, the subspace X_k, Y_k and Z_k are defined by (2). If for each $k = 1, 2, \cdots$, there exist $\rho_k > \gamma_k > 0$ such that*

(A_1) $\alpha_k := \inf\{\varphi(u) \mid u \in Z_k, \|u\| = \gamma_k\} \to \infty$ $(k \to \infty)$;

(A_2) $\beta_k := \max\{\varphi(u) | \ u \in Y_k, \|u\| = \rho_k\} \leq 0$.

then φ has a sequence of critical values tending to $+\infty$.

Proposition 3.9. (Dual Fountain theorem, see [53]) Assume X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$ is an even functional, the subspace X_k, Y_k and Z_k are defined by (2), and there is a $k_0 > 0$ such that, for each $k \geq k_0$, there exists $\rho_k > \gamma_k > 0$ such that

(D_1) $\inf\{\varphi(u) \mid u \in Z_k, \|u\| = \rho_k\} \geq 0$,

(D_2) $\zeta_k := \max\{\varphi(u) | \ u \in Y_k, \|u\| = \gamma_k\} < 0$,

(D_3) $\eta_k := \inf\{\varphi(u) \mid u \in Z_k, \|u\| \leq \rho_k\} \to 0$ $(k \to \infty)$,

(D_4) φ satisfies $(PS)^*_c$ condition for every $c \in [\eta_{k_0}, 0)$,

then φ has a sequence of critical values c^k tending to 0. Moreover, $c^k \in [\eta_k, \zeta_k]$.

4. Main Results and Proofs

In this section, we study the existence of infinitely many pairs of solutions via Fountain Theorem and the dual Fountain Theorem stated before, respectively.

Definition 4.1 . We say $u \in X$ is a weak solution of (P) provided

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v dx + \int_\Omega |u|^{p(x)-2} u \cdot v dx = \int_\Omega f(x,u) v dx, \forall v \in X.$$

It is easy to see that the critical points of φ correspond to the weak solutions of (P).

Remark 1. Regarding the p-Laplacian equations with the critical Sobolev growth conditions, there are many results showing that c is a critical value of φ, when the value c is less than some real number c_∞ which dependents on the best imbedding constant

$$S_p := \inf\{\int_\Omega |\nabla u|^p \, dx | \int_\Omega |u|^{p^*} dx = 1\} = \left(\inf_{u \in W^{1,p}(\Omega)\backslash\{0\}} \frac{|\nabla u|_p}{|u|_{p^*}} \right)^p .$$

Due to the non-homogeneity in problems involving $p(x)$-Laplacian, we could only get the best imbedding constant $C = \inf_{u \in X\backslash\{0\}} \frac{|\nabla u|_{p(\cdot)}}{|u|_{p^*(\cdot)}}$. We can see that $\left(\inf_{u \in X\backslash\{0\}} \frac{|\nabla u|_{p(\cdot)}}{|u|_{p^*(\cdot)}} \right)^{p(x)}$ is a function dependent on the variable x. It is difficult to get the similar results.

As an application of Theorem 2.7, we show the existence of infinitely many pairs of solutions by a perturbation argument.

Theorem 4.2. *If* $f(x,t) = \mu |t|^{\alpha(x)-2} t + \lambda |t|^{q(x)-2} t, \forall (x,t) \in \overline{\Omega} \times \mathbb{R}$, *satisfying* $1 < q(x) \le p^*(x)$, $\alpha^+ < p^-$ *and*

$$q^- > p^+, \tag{7}$$

and the positive parameters λ *and* μ *satisfy one of the following conditions*

(1^0) λ *is fixed, and* μ *is small enough;*

(2^0) $\frac{q^+}{q^+-p^-} > \frac{q^-}{q^--\alpha^+}$, μ *is fixed and* λ *is small enough;*

(3^0) $\frac{q^+}{q^+-p^-} < \frac{q^-}{q^--\alpha^+}$, $\lambda \to 0^+$ *and* $\mu \to +\infty$ *such that* $(\frac{1}{\lambda})^{\frac{q^-}{q^--\alpha^+} - \frac{q^+}{q^+-p^-}} \mu^{\frac{q^-}{q^--\alpha^+}}$ *is small enough;*

(4^0) $\lambda \to 0^+$ *and* $\mu \to 0^+$ *such that* $(\frac{1}{\lambda})^{\frac{q^-}{q^--\alpha^+} - \frac{q^+}{q^+-p^-}} \mu^{\frac{q^+}{q^+-\alpha^-}}$ *is small enough;*

(5^0) $\lambda \to +\infty$ *and* $\mu \to 0^+$ *such that* $(\frac{1}{\lambda})^{\frac{q^+}{q^+-\alpha^-} - \frac{q^-}{q^--p^+}} \mu^{\frac{q^+}{q^+-\alpha^-}}$ *is small enough;*

then (P) has a sequence of pairs of solutions $\{\pm u_n\}$ *such that* $\varphi(\pm u_n) < 0$ *and* $\varphi(\pm u_n) \to 0$.

Proof. Let's consider a sequence of perturbation problems as follows.

$$(\mathrm{P}_n) \qquad \begin{cases} -div(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = f_n(x,u) & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where

$$f_n(x,t) = \mu\,|t|^{\alpha(x)-2}t + \lambda\,|t|^{q(x)-2-\varepsilon_n}t, \forall (x,t) \in \overline{\Omega}\times\mathbb{R},$$

where ε_n is decreasing, $\varepsilon_1 < q^- - p^+$ and $\varepsilon_n \to 0^+$. The corresponding functional of (P_n) is

$$\varphi_n(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_\Omega F_n(x,u)dx,$$

where $F_n(x,u) = \int_0^u f_n(x,t)dt$.

For the functionals φ_n, we will check the conditions of Proposition 3.9 item by item.

Assume $\rho_k = 1$ and k is large enough. For any $u \in Z_k$ with $\|u\|_{p(\cdot)} = \rho_k$, it follows from $q^- > p^+$, Theorem 3.2 and Lemma 3.6 that

$$\varphi_n(u) \geq C_k > 0.$$

Hence (D_1) is satisfied.

Assume $\gamma_k < \rho_k$ is small enough. For any $u \in Y_k$ with $\|u\|_{p(\cdot)} = \gamma_k$, we have

$$\varphi_n(u) \leq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx - \int_\Omega \frac{\mu\,|u|^{\alpha(x)}}{\alpha(x)}dx \triangleq b_k < 0.$$

So (D_2) is satisfied.

When $\| u \|_{p(\cdot)}$ is small enough, by direct computations, we have

$$\begin{aligned} \varphi_n(u) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx \\ &\quad - \int_\Omega \frac{\mu\,|u|^{\alpha(x)}}{\alpha(x)}dx - \int_\Omega \frac{\lambda}{q(x)-\varepsilon_n}|u|^{q(x)-\varepsilon_n}dx \\ &\geq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx \\ &\quad - \int_\Omega \frac{\mu\,|u|^{\alpha(x)}}{\alpha(x)}dx - \lambda\left|\frac{1}{q(x)-\varepsilon_n}\right|_{\frac{q(x)}{\varepsilon_n}}|u|_{q(\cdot)}^{q(\zeta)-\varepsilon_n} \\ &\geq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx \\ &\quad - \int_\Omega \frac{\mu\,|u|^{\alpha(x)}}{\alpha(x)}dx - C\lambda\|u\|_{p(\cdot)}^{q(\zeta)-\varepsilon_n} \end{aligned}$$

$$\geq \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx + \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx$$
$$- \int_\Omega \frac{\mu\, |u|^{\alpha(x)}}{\alpha(x)} dx - C\lambda \|u\|_{p(\cdot)}^{q(\zeta)-\varepsilon_1}.$$

Noting that $q^- > p^+$, when k is large enough and $\rho_k = 1$, $\forall u \in Z_k$, $\|u\|_{p(\cdot)} \leq \rho_k$, we have

$$\varphi_n(u) \geq - \int_\Omega \frac{\mu\, |u|^{\alpha(x)}}{\alpha(x)} dx \geq -2C\mu \|u\|_{p(\cdot)} \beta_k(1) \geq -2C\mu\beta_k(1) \triangleq d_k \to 0 \text{ as } k \to \infty,$$

where $\beta_k(1)$ is defined in (6) with $\Theta(u) = \int_\Omega \frac{|u|^{\alpha(x)}}{\alpha(x)} dx$. Hence ($D_3$) is satisfied.

Similarly to the proof of Lemma 3.4, we see that every $(PS)_c^*$ sequence is bounded. Similarly to the proof of the Theorem 4.6 of [55], we see that (D_4) is satisfied.

For any $n = 1, 2, \cdots$, from Proposition 3.9, we see that φ_n has a sequence of critical values $c_n^k \in [d_k, b_k]$, and for every c_n^k, φ_n has a related critical point u_n^k.

If $u_n^k \to u^k \in X$ as $n \to +\infty$, then it is easy to see that u^k is a critical point of φ, and the critical value of $\varphi(u^k) = c^k = \lim_{n\to\infty} c_n^k \in [d_k, b_k]$. Thus u^k is a nontrivial solution to (P). Since $d_k \to 0^-$ as $k \to +\infty$, we can see that φ has infinitely many solutions u^k such that $\varphi(u^k) \to 0^-$ as $k \to +\infty$.

It only remains to prove that $u_n^k \to u^k$ as $n \to \infty$ when k is large enough.

We only need to prove that $u_n^{k_0} \to \omega$ in X and the rest is completely similar.

Since $u_n^{k_0}$ is a critical point of φ_n, we have

$$c_n^{k_0} = J(u_n^{k_0}) - \int_\Omega \frac{1}{q(x)-\varepsilon_n} \lambda \left| u_n^{k_0} \right|^{q(x)-\varepsilon_n} dx - \int_\Omega \frac{1}{\alpha(x)} \mu \left| u_n^{k_0} \right|^{\alpha(x)} dx, \tag{8}$$

and

$$\int_\Omega \left| \nabla u_n^{k_0} \right|^{p(x)} dx + \int_\Omega \left| u_n^{k_0} \right|^{p(x)} dx = \int_\Omega \lambda \left| u_n^{k_0} \right|^{q(x)-\varepsilon_n} dx + \int_\Omega \mu \left| u_n^{k_0} \right|^{\alpha(x)} dx. \tag{9}$$

It follows from (8) and (9) that

$$c_n^{k_0} = \left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n}\right)\lambda \int_\Omega \left| u_n^{k_0} \right|^{q(x)-\varepsilon_n} dx + \left(\frac{1}{p(\xi_1)} - \frac{1}{\alpha(\xi_3)}\right)\mu \int_\Omega \left| u_n^{k_0} \right|^{\alpha(x)} dx,$$

and then

$$\begin{aligned}
\left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n}\right)\lambda \left| u_n^{k_0} \right|_{q(\cdot)-\varepsilon_n}^{q(\xi_4)-\varepsilon_n} &= \left(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n}\right)\lambda \int_\Omega \left| u_n^{k_0} \right|^{q(x)-\varepsilon_n} dx \\
&= c_n^{k_0} + \left(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)}\right)\mu \int_\Omega \left| u_n^{k_0} \right|^{\alpha(x)} dx \\
&\leq c_n^{k_0} + \left(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)}\right)\mu c_1 \left| u_n^{k_0} \right|_{q(\cdot)-\varepsilon_n}^{\alpha(\xi_5)},
\end{aligned}$$

where $\xi_i \in \overline{\Omega}$, $i = 1, \cdots, 5$, $c_1 = 2\sup_n |1|_{\frac{q(\cdot)-\varepsilon_n}{q(\cdot)-\varepsilon_n-\alpha(\cdot)}} < +\infty$. Thus

$$(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n})\lambda \left|u_n^{k_0}\right|_{q(\cdot)-\varepsilon_n}^{q(\xi_4)-\varepsilon_n} \le (\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)})\mu c_1 \left|u_n^{k_0}\right|_{q(\cdot)-\varepsilon_n}^{\alpha(\xi_5)}$$

when n is large enough.

Therefore

$$\left|u_n^{k_0}\right|_{q(\cdot)-\varepsilon_n} \le \Big[\frac{\mu(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)})c_1}{\lambda(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n})}\Big]^{\frac{1}{q(\xi_4)-\alpha(\xi_5)-\varepsilon_n}}, \tag{10}$$

which implies that

$$(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n})\lambda \int_\Omega \left|u_n^{k_0}\right|^{q(x)-\varepsilon_n} dx = (\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n})\lambda \left|u_n^{k_0}\right|_{q(\cdot)-\varepsilon_n}^{q(\xi_4)-\varepsilon_n}$$
$$\le (\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n})\Big[\frac{(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)})c_1}{\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n}}\Big]^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} (\frac{1}{\lambda})^{\frac{\alpha(\xi_5)}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} \mu^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}}.$$

Thus,

$$c_n^{k_0} + (\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)})\mu \int_\Omega \left|u_n^{k_0}\right|^{\alpha(x)} dx$$
$$= (\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n})\lambda \int_\Omega \left|u_n^{k_0}\right|^{q(x)-\varepsilon_n} dx$$
$$\le (\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n})\Big[\frac{(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)})c_1}{\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n}}\Big]^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} (\frac{1}{\lambda})^{\frac{\alpha(\xi_5)}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} \mu^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}}.$$

Then we have

$$J_n(u_n^{k_0}) = c_n^{k_0} + \int_\Omega \frac{1}{q(x)-\varepsilon_n}\lambda \left|u_n^{k_0}\right|^{q(x)-\varepsilon_n} dx + \int_\Omega \frac{\mu}{\alpha(x)} \left|u_n^{k_0}\right|^{\alpha(x)} dx$$
$$\le C\{2(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)})\Big[\frac{(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)})c_1}{\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n}}\Big]^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}}$$
$$(\frac{1}{\lambda})^{\frac{\alpha(\xi_5)}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} \mu^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} + |d_{k_0}|\}.$$

It implies that $\{u_n^{k_0}\}$ is bounded.

From Theorem 2.7, we have

$$u_n^{k_0} \rightharpoonup \omega \text{ in } W_0^{1,p(\cdot)}(\Omega),\ \left|\nabla u_n^{k_0}\right|^{p(x)} \overset{*}{\rightharpoonup} \mu \text{ in } \mathbb{M}(\overline{\Omega}),\ \left|u_n^{k_0}\right|^{q(x)-\varepsilon_n} \overset{*}{\rightharpoonup} v^{\#} \text{ in } \mathbb{M}(\overline{\Omega}).$$

Set

$$C_q^* = \sup\{|\omega_\varepsilon|_{q(\cdot)}^{q+} + 1 \left| \omega_\varepsilon \in W_0^{1,p(\cdot)}(\Omega), |\nabla \omega_\varepsilon|_{p(\cdot)} \le 1\right.\}$$

and $0 < C_q^* < +\infty$. The limit measures are of the form

$$\begin{aligned}\mu &= |\nabla\omega|^{p(x)} + \sum_{j\in J}\mu_j\delta_{x_j} + \widetilde{\mu}, \mu(\overline{\Omega}) \le 1,\\ \upsilon^{\#} &= |\omega|^{p^*(x)} + \sum_{j\in J}\upsilon_j^{\#}\delta_{x_j}, \upsilon^{\#}(\overline{\Omega}) \le C_q^* + |\Omega|,\end{aligned}$$

where $x_j \in \overline{\Omega}$, J is a countable set, $\widetilde{\mu} \in M(\overline{\Omega})$ is a nonatomic measure.

Similarly to the proof of Lemma 3.5, we have

$$\mu_j \le \lambda\upsilon_j^{\#} \le \lambda\upsilon_j \le \lambda C_q^* \max\{\mu_j^{\frac{q^+}{p^-}}, \mu_j^{\frac{q^-}{p^+}}\},$$

and then

$$\upsilon_j^{\#} \ge \left|\lambda(C_q^*)^{\frac{p^-}{q^+}}\right|^{\frac{-1}{1-\frac{p^-}{q^+}}}, \text{ when } \mu(\{x_j\}) \ge 1,$$

$$\upsilon_j^{\#} \ge \left|\lambda(C_q^*)^{\frac{p^+}{q^-}}\right|^{\frac{-1}{1-\frac{p^+}{q^-}}}, \text{ when } 0 < \mu(\{x_j\}) < 1,$$

which implies that there exist only finite $\upsilon_j^{\#} \ne 0$. Without loss of generality, we may assume that $\upsilon_j^{\#} > 0$ for exact $j = 1, \cdots, k$, and then

$$\sum_{j\in J}\upsilon_j^{\#} = \sum_{\mu(\{x_j\})\ge 1}\upsilon_j^{\#} + \sum_{0<\mu(\{x_j\})<1}\upsilon_j^{\#} \ge \sum_{\mu(\{x_j\})\ge 1} a(\frac{1}{\lambda})^{\frac{q^+}{q^+-p^-}} + \sum_{0<\mu(\{x_j\})<1} a(\frac{1}{\lambda})^{\frac{q^-}{q^--p^+}}.$$

Therefore

$$\sum_{j\in J}\upsilon_j^{\#} \ge \begin{cases} b(\frac{1}{\lambda})^{\frac{q^+}{q^+-p^-}}, \lambda < 1\\ b(\frac{1}{\lambda})^{\frac{q^-}{q^--p^+}}, \lambda \ge 1\end{cases}, \tag{11}$$

where b is a constant which is independent on (n, λ, μ).

On the other hand,

$$\int_\Omega \left|u_n^{k_0}\right|^{q(x)-\varepsilon_n} dx = \left|u_n^{k_0}\right|_{q(\cdot)-\varepsilon_n}^{q(\xi_4)-\varepsilon_n} \le \left[\frac{(\frac{1}{\alpha(\xi_3)} - \frac{1}{p(\xi_1)})c_1}{\lambda(\frac{1}{p(\xi_1)} - \frac{1}{q(\xi_2)-\varepsilon_n})}\right]^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}} \mu^{\frac{q(\xi_4)-\varepsilon_n}{q(\xi_4)-\varepsilon_n-\alpha(\xi_5)}},$$

which implies that

$$\sum_{j=1}^{k}\upsilon_j^{\#} \le \lim_{n\to\infty}\int_\Omega \left|u_n^{k_0}\right|^{q(x)-\varepsilon_n} dx \le c_{\#}(\frac{1}{\lambda})^{\frac{q(\xi_4)}{q(\xi_4)-\alpha(\xi_5)}} \mu^{\frac{q(\xi_4)}{q(\xi_4)-\alpha(\xi_5)}}, \tag{12}$$

where $c_{\#}$ is a constant independent on (λ, μ, n).

From (11) and (12), we have

$$\left.\begin{array}{l} b(\frac{1}{\lambda})^{\frac{q^+}{q^+-p^-}}, \lambda < 1 \\ b(\frac{1}{\lambda})^{\frac{q^-}{q^--p^+}}, \lambda \geq 1 \end{array}\right\} \leq \sum_{j=1}^{k} \upsilon_j^{\#} \leq \left\{\begin{array}{l} c_{\#}(\frac{1}{\lambda})^{\frac{q^-}{q^--\alpha^+}} \mu^{\frac{q^+}{q^+-\alpha^-}}, \lambda < 1, \mu < 1, \\ c_{\#}(\frac{1}{\lambda})^{\frac{q^-}{q^--\alpha^+}} \mu^{\frac{q^-}{q^--\alpha^+}}, \lambda < 1, \mu \geq 1, \\ c_{\#}(\frac{1}{\lambda})^{\frac{q^+}{q^+-\alpha^-}} \mu^{\frac{q^+}{q^+-\alpha^-}}, \lambda \geq 1, \mu < 1, \\ c_{\#}(\frac{1}{\lambda})^{\frac{q^+}{q^+-\alpha^-}} \mu^{\frac{q^-}{q^--\alpha^+}}, \lambda \geq 1, \mu \geq 1, \end{array}\right. .$$

Under one of the conditions of (1^0)-(5^0), we get a contradiction.

Thus $\upsilon_j^{\#} = \mu_j = 0$ for $j = 1, 2, \cdots$. Therefore

$$\lim_{n\to\infty} \int_\Omega \left|u_n^{k_0}\right|^{q(x)-\varepsilon_n} dx = \int_\Omega |\omega|^{q(x)}\, dx.$$

Obviously, $\left|u_n^{k_0}\right|^{q(x)-\varepsilon_n} \to |\omega|^{q(x)}$ a.e. on Ω. Thus $\{|u_n|^{q(x)-\varepsilon_n}\}$ is uniformly integrable. It is easy to see that

$$\left|u_n^{k_0}\right|^{q(x)-\frac{q(x)}{q(x)-1}\varepsilon_n} \leq 1 + \left|u_n^{k_0}\right|^{q(x)-\varepsilon_n}.$$

Then $\{\left|u_n^{k_0}\right|^{q(x)-\frac{q(x)}{q(x)-1}\varepsilon_n}\}$ is uniformly integrable. Obviously, $\left|u_n^{k_0}\right|^{q(x)-\varepsilon_n-2} u_n^{k_0} \to |\omega|^{q(x)-2}\,\omega$ a.e. on Ω, then

$$\lim_{n\to\infty} \int_\Omega \left|\left|u_n^{k_0}\right|^{q(x)-\varepsilon_n-2} u_n^{k_0} - |\omega|^{q(x)-2}\,\omega\right|^{\frac{q(x)}{q(x)-1}} dx = 0.$$

It means that $\Psi_n'(u_n^{k_0}) \to \Psi'(\omega)$ in X^*, where $\Psi_n(u_n^{k_0}) = \int_\Omega F_n(x, u_n^{k_0})dx$. Thus $u_n^{k_0} \to L^{-1}(\Psi'(\omega))$. ∎

Remark 2. In [10], Cao and Yan dealt with the existence of infinitely many solutions of the following Laplacian equation Dirichlet problems involving critical nonlinearities and Hardy potential

$$\text{(I)} \qquad \left\{\begin{array}{c} -\triangle u - \frac{\mu}{|x|^2}u = |u|^{2^*-2} u + au \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{array}\right.$$

by considering the following perturbed problem which is of subcritical growth,

$$(\text{I}_n) \qquad \left\{\begin{array}{c} -\triangle u - \frac{\mu}{|x|^2}u = |u|^{2^*-2-\varepsilon_n} u + au \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{array}\right.$$

By applying a local Pohozaev identity, Cao and Yan proved that the solution $\{u_n\}$ of (I_n) converges to a solution u of (I). Because of the nonhomogeneity of $p(x)$-Laplacian, the Pohozaev identity cann't be obtained by the usual methods. So the method in [10] is very hard to be directly used in dealing with the $p(x)$-Laplacian problems.

Assume that

$$f(x,t) = g(x,t) + h(x)\,|t|^{p^*(x)-2}\,t, \forall (x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{13}$$

Denote

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx + \int_\Omega \frac{1}{p(x)}\,|u|^{p(x)}\,dx - \int_\Omega G(x,u)dx.$$

where $G(x,t) = \int_0^t g(x,s)ds$.

Theorem 4.3. *Assume that* (*13*) *is satisfied, and* (*P*) *satisfies the following conditions* (F_1)*-*(F_5)*:*

(F_1) Ω *is a radially symmetric domain with respect to the origin,* $1 < p < N$ *and the function* p *is radial, i.e.*

$$p(x) = p(|x|) \textit{ for any } x \in \overline{\Omega},$$

(F_2) g *and* h *are radial with respect to the space variable* x*, i.e.*

$$g(x,t) = g(|x|\,,t) \textit{ and } h(x) = h(|x|) \textit{ for any } (x,t) \in \overline{\Omega} \times \mathbb{R},$$

(F_3) $f(x,-t) = -f(x,t)$ *for any* $(x,t) \in \overline{\Omega} \times \mathbb{R}$,

(F_4) $|g(x,t)| \leq C(1 + |t|^{\theta(x)-1})$ *for any* $(x,t) \in \overline{\Omega} \times \mathbb{R}$*, where the function* $\theta(\cdot) \in C^1(\overline{\Omega})$ *satisfies* $p(x) < \theta(x) < p^*(x)$*,*

(F_5) $h(0) = 0$*,* $h \in C(\overline{\Omega})$ *and* $[h(\cdot)]^{\frac{-\theta(\cdot)}{p^*(\cdot)-\theta(\cdot)}} \in L^1(\Omega)$.

Then (P) has a sequence of radial solutions $\{\pm u_m\}$ such that $\varphi(\pm u_m) \to +\infty$.

Proof. First, we will prove that φ satisfies the (PS) condition.

Let $\{u_n\}$ be a (PS) sequence. We claim that $\{u_n\}$ is bounded. Similarly to the proof of Lemma 3.4, we have

$$\begin{aligned}
&c + \|u_n\|_{p(\cdot)} \\
&\geq \varphi(u_n) - (\varphi'(u_n), \frac{1}{\theta(x)}u_n) \\
&= \int_\Omega \frac{1}{p(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx \\
&\quad - \int_\Omega F(x,u_n)dx - \int_\Omega \frac{1}{\theta(x)}(|\nabla u_n|^{p(x)} + |u_n|^{p(x)})dx \\
&\quad + \int_\Omega \frac{1}{\theta(x)}u_n f(x,u_n)dx + \int_\Omega \frac{1}{\theta^2(x)}u_n\,|\nabla u_n|^{p(x)-2}\,\nabla u_n \nabla\theta(x)dx
\end{aligned}$$

$$\begin{aligned}
&\geq \int_\Omega (\frac{1}{p(x)}-\frac{1}{\theta(x)})(|\nabla u_n|^{p(x)}+|u_n|^{p(x)})dx+\int_\Omega (\frac{1}{\theta(x)}-\frac{1}{p^*(x)})h(x)\,|u_n|^{p^*(x)}\,dx\\
&\quad -\int_\Omega \frac{|\nabla\theta(x)|}{\theta^2(x)}\,|u_n|\,|\nabla u_n|^{p(x)-1}\,dx-2C_1\int_\Omega |u_n|^{\theta(x)}\,dx-C_2\\
&\geq l_1\int_\Omega (|\nabla u_n|^{p(x)}+|u_n|^{p(x)})dx+\int_\Omega (\frac{1}{\theta(x)}-\frac{1}{p^*(x)})h(x)\,|u_n|^{p^*(x)}\,dx\\
&\quad -\int_\Omega \frac{|\nabla\theta(x)|}{\theta^2(x)}\,|u_n|\,|\nabla u_n|^{p(x)-1}\,dx-2C_1\int_\Omega |u_n|^{\theta(x)}\,dx-C_2,
\end{aligned}$$

where $l_1=\inf\limits_{x\in\Omega}(\frac{1}{p(x)}-\frac{1}{\theta(x)})$.

Since $[h(\cdot)]^{\frac{-\theta(\cdot)}{p^*(\cdot)-\theta(\cdot)}}\in L^1(\Omega)$ and $\theta\in C^1(\overline{\Omega})$, similarly to the proof of Lemma 3.4, we have

$$\begin{aligned}
&\frac{|\nabla\theta(x)|}{\theta^2(x)}\,|u_n|\,|\nabla u_n|^{p(x)-1}\\
&\leq C_3\frac{1}{p(x)}\frac{1}{\varepsilon_1^{p(x)}}\{\frac{p^*(x)-p(x)}{p^*(x)}\varepsilon_1^{\frac{-p^*(x)p(x)}{p^*(x)-p(x)}}[h(x)]^{\frac{-p(x)}{p^*(x)-p(x)}}+\frac{p(x)}{p^*(x)}\varepsilon_1^{p^*(x)}h(x)\,|u_n|^{p^*(x)}\}\\
&\quad +C_3\frac{p(x)-1}{p(x)}\varepsilon_1^{\frac{p(x)}{p(x)-1}}\,|\nabla u_n|^{p(x)},
\end{aligned}$$

and

$$\begin{aligned}
|u_n|^{\theta(x)}&\leq\frac{p^*(x)-\theta(x)}{p^*(x)}\{\frac{1}{\varepsilon_1}[h(x)]^{\frac{-\theta(x)}{p^*(x)}}\}^{\frac{p^*(x)}{p^*(x)-\theta(x)}}+\frac{\theta(x)}{p^*(x)}\{\varepsilon_1[h(x)]^{\frac{\theta(x)}{p^*(x)}}\,|u_n|^{\theta(x)}\}^{\frac{p^*(x)}{\theta(x)}}\\
&=\frac{p^*(x)-\theta(x)}{p^*(x)}(\frac{1}{\varepsilon_1})^{\frac{p^*(x)}{p^*(x)-\theta(x)}}[h(x)]^{\frac{-\theta(x)}{p^*(x)-\theta(x)}}+\frac{\theta(x)}{p^*(x)}(\varepsilon_1)^{\frac{p^*(x)}{\theta(x)}}h(x)\,|u_n|^{p^*(x)}.
\end{aligned}$$

Noting that $\frac{\theta(x)}{p^*(x)-\theta(x)}>\frac{p(x)}{p^*(x)-p(x)}$, when positive constants ε_1 is small enough, we have

$$c+\|u_n\|_{p(\cdot)}\geq\varphi(u_n)-(\varphi'(u_n),\frac{1}{\theta(x)}u_n)\geq\frac{2l_1}{3}\int_\Omega(|\nabla u_n|^{p(x)}+|u_n|^{p(x)})dx-C_4.$$

Thus $\{u_n\}$ is bounded. Consequently,

$$u_n\rightharpoonup u \text{ in } W_0^{1,p(\cdot)}(\Omega),\ |\nabla u_n|^{p(x)}\overset{*}{\rightharpoonup}\mu \text{ in } \mathbb{M}(\overline{\Omega}),\ |u_n|^{p^*(x)}\overset{*}{\rightharpoonup}\upsilon \text{ in } \mathbb{M}(\overline{\Omega}).$$

Set

$$C_{p^*}^*=\sup\{|\omega_\varepsilon|_{p^*(\cdot)}^{p^{*+}}+1\,\Big|\,\omega_\varepsilon\in W_0^{1,p(\cdot)}(\Omega),|\nabla\omega_\varepsilon|_{p(\cdot)}\leq 1\}$$

and $0 < C_{p^*}^* < +\infty$. According to Proposition 2.6, the limit measures are of the form

$$\begin{aligned} \mu &= |\nabla u|^{p(x)} + \sum_{j\in J}\mu_j\delta_{x_j} + \widetilde{\mu}, \mu(\overline{\Omega}) \leq 1, \\ \upsilon &= |u|^{p^*(x)} + \sum_{j\in J}\upsilon_j\delta_{x_j}, \upsilon(\overline{\Omega}) \leq C^*, \end{aligned}$$

where $x_j \in \overline{\Omega}$, and J is a countable set, and $\widetilde{\mu} \in M(\overline{\Omega})$ is a nonatomic positive measure.

Similarly to the proof of Lemma 3.5, we may assume that $\mu_j > 0$ for $j = 1, \cdots, k$, and the rest are zero.

If $x_k \neq 0$, since (P) is radial, it is easy to see that for any $x \in \overline{\Omega}$ such that $|x| = |x_k|$ and $x \notin \{x_1, \cdots, x_k\}$, we have $\mu(x) = \mu(x_k)$. It is a contradiction. Thus $\mu(x) = 0$ for any $x \neq 0$.

For every $\varepsilon > 0$, we set $\phi_\varepsilon(x) = \phi(\frac{x}{\varepsilon}), x \in \overline{\Omega}$, where $\phi \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \phi \leq 1$, $\phi \equiv 1$ in $B_1\{0\}$ and $\phi \equiv 0$ in $\mathbb{R}^N \backslash B_2\{0\}$ and $|\nabla\phi| \leq 2$. Since $\varphi'(u_n) \to 0$ in X^* as $n \longrightarrow \infty$ and $\{u_n\}$ is bounded, we have $(\varphi'(u_n), \phi_\varepsilon u_n) \to 0$, and then

$$\begin{aligned} (\varphi'(u_n), \phi_\varepsilon u_n) &= \int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla(\phi_\varepsilon u_n)dx \\ &\quad + \int_\Omega |u_n|^{p(x)-2} u_n\phi_\varepsilon u_n dx - \int_\Omega f(x,u_n)\phi_\varepsilon u_n dx \\ &= \int_\Omega |\nabla u_n|^{p(x)}\phi_\varepsilon dx + \int_\Omega |u_n|^{p(x)}\phi_\varepsilon dx \\ &\quad + \int_\Omega u_n |\nabla u_n|^{p(x)-2}\nabla u_n \cdot \nabla\phi_\varepsilon dx - \int_\Omega f(x,u_n)\phi_\varepsilon u_n dx \to 0. \end{aligned}$$

Without loss of generality, we may assume that $u_n \rightharpoonup u$ in $W_0^{1,p(\cdot)}(\Omega)$. From (5), we have

$$\int_\Omega \phi_\varepsilon d\mu + \int_\Omega |u|^{p(x)}\phi_\varepsilon dx + \int_\Omega uT\cdot\nabla\phi_\varepsilon dx - \int_\Omega g(x,u)\phi_\varepsilon u dx - \int_\Omega h(x)\phi_\varepsilon d\upsilon = 0. \tag{14}$$

It is easy to see that

$$\left|\int_\Omega uT\cdot\nabla\phi_\varepsilon dx\right| \leq |T|_{\frac{p(\cdot)}{p(\cdot)-1}} |u\nabla\phi_\varepsilon|_{p(\cdot)},$$

$$\int_\Omega \left||\nabla\phi_\varepsilon|^{p(x)}\right|^{(\frac{p^*(x)}{p(x)})^0} dx = \int_{B(0,2\varepsilon)} |\nabla\phi_\varepsilon|^N dx \leq (\frac{2}{\varepsilon})^N meas B(0,2\varepsilon) = \frac{4^N}{N}\varpi_N,$$

where ϖ_N is the Hausdorff measure of the unit ball of $\mathbb{R}^N$, and

$$\int_\Omega |u\nabla\phi_\varepsilon|^{p(x)} dx = \int_{B(0,2\varepsilon)} |u\nabla\phi_\varepsilon|^{p(x)} dx \leq 2\left||\nabla\phi_\varepsilon|^{p(x)}\right|_{(\frac{p^*(\cdot)}{p(\cdot)})^0} \left||u|^{p(x)}\right|_{\frac{p^*(\cdot)}{p(\cdot)},B(0,2\varepsilon)}.$$

Then, we have

$$\left|\int_{\Omega} uT\cdot\nabla\phi_\varepsilon dx\right| \to 0 \text{ as } \varepsilon\to 0. \tag{15}$$

It follows from (14) and (15) that

$$\mu(0)-h(0)\upsilon(0)=0. \tag{16}$$

Combining (F_2), (F_5) and (16), we have $\mu(0)=0$. Thus $\mu(x)=0$ for any $x\in\overline{\Omega}$. Proposition 2.6 guaranties that $\upsilon(x)=0$ for any $x\in\overline{\Omega}$. Thus $\upsilon_j=\mu_j=0$ for $j=1,2,\cdots$. Thus

$$\lim_{n\to\infty}\int_{\Omega}|u_n|^{p^*(x)}\,dx=\int_{\Omega}|u|^{p^*(x)}\,dx.$$

Noting that we also have $u_n\to u$ a.e. in Ω, we conclude that $\{|u_n|^{p^*(\cdot)}\}$ is uniformly integrable. Thus $\{|u_n-u|^{p^*(\cdot)}\}$ is uniformly integrable. According to the Vitally Theorem, we have

$$\lim_{n\to\infty}\int_{\Omega}|u_n-u|^{p^*(x)}\,dx=\int_{\Omega}\lim_{n\to\infty}|u_n-u|^{p^*(x)}\,dx=0.$$

It means that $\Psi'(u_n)\to\Psi'(u)$ in X^*. Then $u_n\to u_1^*:=L^{-1}(\Psi'(u))$. Therefore, $u_n\to u$ in $W_0^{1,p(\cdot)}(\Omega)$. So φ satisfies the (PS) condition.

Next, we will prove that $\Psi(\cdot)$ is weak-strong continuous.

If $u_n\rightharpoonup u$, we only need to prove

$$\lim_{n\to\infty}\int_{\Omega}h(x)\,|u_n|^{p^*(x)}\,dx=\int_{\Omega}h(x)\,|u|^{p^*(x)}\,dx. \tag{17}$$

Obviously, $\{u_n\}$ is bounded in X. From the above proof, we see that $\upsilon\{x\}=0$ when $x\neq 0$. Therefore, for any $\delta>0$, we have

$$\int_{\Omega\backslash\overline{B(0,\delta)}}\left|h(x)\,|u_n|^{p^*(x)}-h(x)\,|u|^{p^*(x)}\right|dx\to 0 \text{ as } n\to\infty.$$

For any $\varepsilon>0$, it follows from the boundedness of $\{u_n\}$ that

$$\begin{aligned}
&\int_{\Omega}\left|h(x)\,|u_n|^{p^*(x)}-h(x)\,|u|^{p^*(x)}\right|dx\\
&=\int_{\Omega\backslash\overline{B(0,\delta)}}\left|h(x)\,|u_n|^{p^*(x)}-h(x)\,|u|^{p^*(x)}\right|dx+\int_{\overline{B(0,\delta)}}\left|h(x)\,|u_n|^{p^*(x)}-h(x)\,|u|^{p^*(x)}\right|dx\\
&\le\max_{x\in\overline{B(0,\delta)}}|h(x)|\int_{\Omega}(|u_n|^{p^*(x)}+|u|^{p^*(x)})dx+\frac{\varepsilon}{2}\le\varepsilon,
\end{aligned}$$

when δ is small enough and n is large enough. Thus (17) is valid.

Similarly to the proof of Theorem 4.3 in [55], we see that the conditions (A_1) and (A_2) of Proposition 3.8 (Fountain Theorem) are satisfied. So φ has a sequence of critical points $\{\pm u_m\}$ in $W_{0,r}^{1,p(\cdot)}(\Omega)$ such that $\varphi(\pm u_m) \to +\infty$, where

$$W_{0,r}^{1,p(\cdot)}(\Omega) = \left\{ u \in W_0^{1,p(\cdot)}(\Omega) \,|u \text{ is radial} \right\},$$

and it is easy to see that $\{\pm u_m\}$ are radial solutions of (P). ∎

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Yuan Liang and Xianbin Wu
Junior College
Zhejiang Wanli University
Ningbo, Zhejiang 315100
P. R. China.
E-mail: wxb3210@zwu.edu.cn

Qihu Zhang
Department of Mathematics and Information Science
Zhengzhou University of Light Industry
Zhengzhou, Henan 450002
P. R. China.
E-mail: zhangqihu@yahoo.com
zhangqh1999@yahoo.com.cn

Chunshan Zhao
Department of Mathematical Sciences
Georgia Southern University
Statesboro, GA 30460
U.S.A.